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Nonlinear Variational Inequalities and Differential Games with Stopping Times

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Differential games with stopping times are introduced. The upper value satisfies a variational inequality of the first order. Next, stochastic differential games with stopping times are introduced and studied using nonlinear parabolic variational inequalities. The behavior of the value is studied as the coefficient of the diffusion matrix goes to zero.

INTRODUCTION

In this paper we consider variational inequalities in a strip $R^m \times (0, T_0)$, of the form

$$\begin{gathered} ((\partial u/\partial t) + F(x, t, u, u_x))(v - u) \leqslant 0 \qquad \text{a.e.} \\ \forall v \mid g_1 \leqslant v \leqslant g_2, \qquad g_1 \leqslant u \leqslant g_2, \end{gathered} \tag{1}$$

$$\begin{gathered} \left(\frac{\partial w}{\partial t} + \sum_{i,j=1}^{m} a_{ij} \frac{\partial^2 w}{\partial x_i \partial x_j} + F(x, t, w, w_x)\right)(v - w) \leqslant 0 \qquad \text{a.e.} \\ \forall v \mid g_1 \leqslant v \leqslant g_2, \qquad g_1 \leqslant w \leqslant g_2 \end{gathered} \tag{2}$$

where (a_{ij}) is a positive definite matrix, $g_2 \geqslant g_1$, and F is a nonlinear function satisfying certain growth conditions. Denote by u_ϵ the solution of (2) when $a_{ij} = \frac{1}{2}\epsilon^2\delta_{ij}$, $g_2 \equiv \infty$.

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It will be shown that the problem (2) has a unique solution. As for (1), it generally has an infinite number of solutions. One of the main results of the present paper is that when $g_2 \equiv \infty$ there exists a solution V^+ of (1) such that

$$u_\epsilon(x, t) \to V^+(x, t) \qquad \text{as} \quad \epsilon \to 0. \tag{3}$$

The solution V^+ is obtained as the upper value of a certain model of differential game with stopping time. Thus in order to prove (3), we first develop the theory of such games. When comparing u_ϵ with V^+, we resort to a method of penalty, i.e., we consider V^+ as a limit of upper values of differential games with increasing penalty (but without stopping times); u_ϵ can also be considered as a limit of nonlinear parabolic equations with penalty. We then apply, for the approximating models with penalty, known results on differential games.

In Section 1 we introduce the model of differential games with stopping times and prove that an upper value V^+ exists. If the controls appear separated in both the dynamics and the payoff, then the value is shown to exist, i.e., $V^+ = V^-$.

In Section 2 we derive for V^+ the variational inequality (1) with F being the upper Hamiltonian function $H^+(x, t, u, p)$.

In Section 3, following Fleming [6] we write a general function $F(x, t, u, p)$ satisfying some growth conditions as $H^+(x, t, u, p)$ of a certain differential game for u and p in some region $|u| \leqslant M$, $|p| \leqslant \mu$ and prove that $|V^+| \leqslant M$, $|\partial V^+/\partial x| \leqslant \mu$. Consequently V^+ is a solution of (1).

In Section 4 we study the problem (2), and prove that it has a unique solution; here we extend the methods of Bensoussan–Lions [1, 2] and Friedman [14, 15], who dealt with the case where $H^+(x, t, u, p)$ is linear in u, p or when there is only a one side inequality ($g_1 \equiv -\infty$).

Section 5 is concerned with an application of the results of Section 4 to stochastic differential games with Markov stopping times and Markov strategies. It generalizes some results of Bensoussan–Lions [1] and Friedman [14, 15] (where no control functions are present).

In Section 6, we prove (3) in the case where there is only one player, i.e., in the case of a control problem. Thus F satisfies rather restrictive conditions and $g_1 \equiv -\infty$. In this case, however, the proof of (3) is extremely simple.

In Section 7 we prove (3) in the general case. We introduce an auxiliary differential game without stopping time, but with a penalty of magnitude $1/\beta$. We rely heavily on the theory of differential games

[4, 13, 16]. There is, however, one new and crucial estimate that is derived here, namely,

$$| V^\delta - V_\beta{}^\delta | \leqslant C\beta \qquad \text{for all} \quad \delta, \tag{6}$$

where V^δ, $V_\beta{}^\delta$ are the upper δ-values for the games with stopping times and with β penalty, respectively. Finally, an application of (3) to differential games is given, namely, a proof of existence of value for a deterministic differential game with stopping times.

1. Differential Games with Stopping Times

Let Y and Z be compact subsets of some Euclidean spaces R^p and R^q respectively. Let $0 < T_0 < +\infty$. Consider a differential system of m equations

$$dx/dt = f(x, t, y, z), \tag{1.1}$$

and assume:

(A) $f(x, t, y, z)$ is continuous in $(x, t, y, z) \in R^m \times [0, T_0] \times Y \times Z$ and

$$x \cdot f(x, t, y, z) \leqslant k(t)(1 + | x |^2), \int_0^{T_0} k(t)\, dt < +\infty;$$

further, for any $R > 0$ there is a function $k_R(t)$ such that

$$| f(x, t, y, z) - f(\bar{x}, t, y, z)| \leqslant k_R(t)| x - \bar{x} |, \int_0^{T_0} k_R(t)\, dt < \infty.$$

A measurable function $y(t)$ with values in Y is called a *control function* for the *player* y. Similarly, a measurable function $z(t)$ with values in Z is called a *control function* for the *player* z. For any initial condition

$$x(\tau) = \xi \qquad (0 \leqslant \tau \leqslant T_0) \tag{1.2}$$

and for any control functions $y = y(t)$, $z = z(t)$ in $[\tau, T_0]$, there exists a unique (absolutely continuous) solution $x(t)$ of (1.1), (1.2).

In the theory of differential games [11], we associate with the system (1.1), (1.2) a payoff

$$P_{\xi\tau}(y, z) = \int_\tau^{T_0} h(x, t, y, z) \exp\left[\int_\tau^t k(x, s, y, z)\, ds\right] dt$$
$$+ g(x(T_0)) \exp\left[\int_\tau^{T_0} k(x, s, y, z)\, ds\right];$$

in fact, one usually deals with the case where $k \equiv 0$, but the theory extends with trivial changes to general k.

In the model that we shall now introduce, the player y is going to choose both a control function $y(t)$ and a *stopping time* S, and the player z is going to choose both a control function $z(t)$ and a *stopping time* T. What this in effect means is that the payoff will be

$$\begin{aligned} P_{\xi\tau}(y, S; z, T) = &\int_\tau^{S\wedge T} h(x, t, y, z) \exp\left[\int_\tau^t k(x, s, y, z)\, ds\right] dt \\ &+ \chi_{S\leqslant T}\, g_1(x(S), S) \exp\left[\int_\tau^S k(x, s, y, z)\, ds\right] \\ &+ \chi_{T<S}\, g_2(x(T), T) \exp\left[\int_\tau^T k(x, s, y, z)\, ds\right] \end{aligned} \tag{1.4}$$

where $S \wedge T = \min(S, T)$ and

$$\chi_{S\leqslant T} = \begin{cases} 1 & \text{if } S \leqslant T \\ 0 & \text{if } S > T \end{cases}, \qquad \chi_{T<S} = \begin{cases} 1 & \text{if } T < S \\ 0 & \text{if } T \geqslant S \end{cases}.$$

The numbers S, T are any numbers in the interval $[\tau, T_0]$.

The player y wishes to maximize the payoff, and the player z wishes to minimize the payoff. We shall assume:

(B) the functions $h(x, t, y, z)$, $k(x, t, y, z)$ and $g_1(x, t)$, $g_2(x, t)$ are continuous in $(x, t, y, z) \in R^m \times [0, T_0] \times Y \times Z$, and

$$g_1(x, t) \leqslant g_2(x, t). \tag{1.5}$$

Analogously to the theory of differential games (when the payoff is given by (1.3)) we introduce a sequence of upper δ-games. Let $\delta = (T_0 - \tau)/N$ where $N = 2^n$, $n = 1, 2, \ldots$, and let $I_j = \{t;\, t_{j-1} < t \leqslant t_j\}$ where $t_j = \tau + j\delta$. Let Y_j be the set of all measurable functions $y_j(t)$ defined on I_j with values in Y. Similarly define Z_j. Let B_j be the set consisting of two functions defined on $I_j : b_j \equiv 0$ and $b_j \equiv 1$. Similarly, let C_j be the set consisting of two functions defined on $I_j : c_j \equiv 0$ and $c_j \equiv 1$. Let

$$\hat{Y}_j = Y_j \times B_j, \qquad \hat{Z}_j = Z_j \times C_j.$$

An upper δ-strategy $\hat{\Gamma}^\delta$ for the player y is a vector $(\hat{\Gamma}^{\delta,1}, \ldots, \hat{\Gamma}^{\delta,N})$ where $\hat{\Gamma}^{\delta,j}$ is a mapping from $\hat{Z}_1 \times \hat{Y}_1 \times \cdots \times \hat{Z}_{j-1} + \hat{Y}_{j-1} \times \hat{Z}_j$ into $\hat{Y}_j$.

A lower δ-strategy $\hat{\Delta}_\delta$ for the player z is a vector $(\hat{\Delta}_{\delta,1}, \ldots, \hat{\Delta}_{\delta,N})$

where $\hat{\Delta}_{\delta,1}$ is an element from $\hat{Z}_1$, and $\hat{\Delta}_{\delta,j}$ is a mapping from $\hat{Z}_1 \times \hat{Y}_1 \times \cdots \times \hat{Z}_{j-1} \times \hat{Y}_{j-1}$ into $\hat{Z}_j$.

Similarly, one defines a lower δ-strategy $\hat{\Gamma}_\delta = (\hat{\Gamma}_{\delta,1},..., \hat{\Gamma}_{\delta,N})$ for y and an upper δ-strategy $\hat{\Delta}^\delta = \hat{\Delta}^{\delta,1},..., \hat{\Delta}^{\delta,N})$ for z.

A vector $(\hat{\Delta}_\delta, \hat{\Gamma}^\delta)$ determines an outcome $\hat{y}$, $\hat{z}$ of controls; $\hat{y}(t) = (y(t), b(t))$, $\hat{z}(t) = (z(t), c(t))$, where $y(t)$, $z(t)$ are control functions. We refer to $\hat{y}(t)$, $\hat{z}(t)$ as *control functions with stopping times*, or, briefly, also as control functions.

Denote by S the first time t_{j-1} such that $b(t) \equiv 1$ on I_j and denote by T the first time t_{l-1} such that $c(t) \equiv 1$ on I_l. Thus, the first interval I_j where $b(t) \equiv 1$ corresponds to y stopping at time t_{j-1}, and the first interval I_l where $c(t) \equiv 1$ corresponds to z stopping at the time t_{l-1}.

We can now correspond to $\hat{y}$, $\hat{z}$ the pairs (y, S) and (z, T) where $y = y(t)$ and $z = z(t)$. We define

$$P_{\xi\tau}[\hat{\Delta}_{\delta,1}, \hat{\Gamma}^{\delta,1},..., \hat{\Delta}_{\delta,N}, \hat{\Gamma}^{\delta,N}] \equiv P_{\xi\tau}[\hat{\Delta}_\delta, \hat{\Gamma}^\delta] \equiv P_{\xi\tau}(\hat{y}, \hat{z}) \equiv P_{\xi\tau}(y, S; z, T)$$

where the right-hand side is given by (1.4). For simplicity, we shall often omit the subindices ξ, τ, i.e., we shall write

$$P[\cdots] = P_{\xi\tau}[\cdots],$$
$$P(\cdots) = P_{\xi\tau}(\cdots).$$

DEFINITIONS. We denote the above scheme by G^δ and call it an ***upper* δ*-game***. The ***upper* δ*-value*** is the number

$$V^\delta(\xi, \tau) = \inf_{\hat{\Delta}_{\delta,1}} \sup_{\hat{\Gamma}^{\delta,1}} \cdots \inf_{\hat{\Delta}_{\delta,N}} \sup_{\hat{\Gamma}^{\delta,N}} P_{\xi\tau}[\hat{\Delta}_{\delta,1}, \hat{\Gamma}^{\delta,1},..., \hat{\Delta}_{\delta,N}, \hat{\Gamma}^{\delta,N}]. \quad (1.6)$$

Similarly we define *lower δ-games* G_δ and the *lower δ-value*

$$V_\delta(\xi, \tau) = \sup_{\hat{\Gamma}_{\delta,1}} \inf_{\hat{\Delta}^{\delta,1}} \cdots \sup_{\hat{\Gamma}_{\delta,N}} \inf_{\hat{\Delta}^{\delta,N}} P_{\xi\tau}[\hat{\Gamma}_{\delta,1}, \hat{\Delta}^{\delta,1},..., \hat{\Gamma}_{\delta,N}, \hat{\Delta}^{\delta,N}]. \quad (1.7)$$

As in [11], one can show that

$$V^\delta(\xi, \tau) = \inf_{\hat{\Delta}_\delta} \sup_{\hat{\Gamma}^\delta} P_{\xi\tau}[\hat{\Delta}_\delta, \hat{\Gamma}^\delta] = \sup_{\hat{\Gamma}^\delta} \inf_{\hat{\Delta}_\delta} P_{\xi\tau}[\hat{\Delta}_\delta, \hat{\Gamma}^\delta].$$

DEFINITIONS. The set $G = \{\{G^\delta\}, \{G_\delta\}\}$ is called the ***differential game with stopping time*** associated with (1.1), (1.2), (1.4). If

$$\lim_{\delta\to 0} V^\delta(\xi, \tau) \qquad (\delta = (T_0 - \tau)/2^n, n \to +\infty)$$

exists, then we say that the game G has an *upper value* $V^+(\xi, \tau)$, where $V^+(\xi, \tau) = \lim_{\delta\to 0} V^\delta(\xi, \tau)$. Similarly, we define the *lower value*

$$V^-(\xi, \tau) = \lim_{\delta\to 0} V_\delta(\xi, \tau).$$

Since clearly $V^\delta(\xi, \tau) \geqslant V_\delta(\xi, \tau)$, we have

$$V^+(\xi, \tau) \geqslant V^-(\xi, \tau) \tag{1.9}$$

when both limits exist. If in (1.9) we have equality, then we say that the game has *value* $V(\xi, \tau)$ where $V(\xi, \tau) = V^+(\xi, \tau) = V^-(\xi, \tau)$.

THEOREM 1.1. *If* (A) *and* (B) *hold, then* $V^+(\xi, \tau)$ *and* $V^-(\xi, \tau)$ *exist.*

Proof. It is enough to prove the existence of $V^+(\xi, \tau)$. Let $\delta = (T_0 - \tau)/2^n$, $\delta' = (T_0 - \tau)/2^{n+1}$, $N = 2^n$. By (1.8), for any $\epsilon > 0$ there exists a lower δ-strategy $\hat{\Delta}_\delta{}^\epsilon$ such that

$$V^\delta + \epsilon \geqslant P[\hat{\Delta}_\delta{}^\epsilon, \hat{\Gamma}^\delta] \qquad \text{for any} \quad \hat{\Gamma}^\delta. \tag{1.10}$$

We shall use $\hat{\Delta}_\delta{}^\epsilon$ in order to construct a "good" δ'-strategy $\hat{\Delta}_{\delta'}$ for z.

Let $I_j{}^1 = (t_{j-1}, t_{j-1} + \delta/2]$, $I_j{}^2 = (t_{j-1} + \delta/2, t_j]$. Denote by $Y_{j,1}$ and $Y_{j,2}$ the spaces of all measurable functions defined on $I_{j,1}$ and $I_{j,2}$ respectively, with values in Y. Similarly, define $B_{j,1}$, $B_{j,2}$, $\hat{Y}_{j,1}$, $\hat{Y}_{j,2}$, $Z_{j,1}$, $Z_{j,2}$, $C_{j,1}$, $C_{j,2}$, $\hat{Z}_{j,1}$, $\hat{Z}_{j,2}$.

Each control $\hat{y}$ which arises from a δ-game can be viewed as a vector

$$(\hat{y}_1, \hat{y}_2, \ldots, \hat{y}_N) \qquad \text{where} \qquad \hat{y}_j = (y_j, b_j) \in \hat{Y}_j. \tag{1.11}$$

One can correspond to it a vector which arises in a δ'-game, namely

$$(\hat{y}_{1,1}, \hat{y}_{1,2}, \hat{y}_{2,1}, \hat{y}_{2,2}, \ldots, \hat{y}_{N,1}, \hat{y}_{N,2}) \text{ where } \hat{y}_{j,i} = (y_{j,i}, b_{j,i}) \in \hat{Y}_{j,i}. \tag{1.12}$$

The converse is not always true. In fact, a control function given by (1.12) can be an outcome of a δ-game if and only if

$$b_{j,1} = b_{j,2} \qquad \text{for} \quad j = 1, 2, \ldots, N. \tag{1.13}$$

The same discussion holds, of course, also for control functions $\hat{z}$.

Let $\hat{y}$ be any control function given by (1.12) (i.e., which arises in a δ'-game). Define

$$b_{j,1}^* = b_{j,2}^* = \begin{cases} 0 & \text{if } j = 1 \quad \text{or if} \quad b_{i,1} = b_{i,2} = 0 \quad \text{for all} \quad 1 \leqslant i < j, \\ 1 & \text{otherwise,} \end{cases}$$

$$y_j{}^* = \begin{cases} y_{j,1} & \text{on} \quad I_{j,1} \\ y_{j,2} & \text{on} \quad I_{j,2} \end{cases}, \quad b_j{}^* = \begin{cases} b_{j,1}^* & \text{on} \quad I_{j,1} \\ b_{j,2}^* & \text{on} \quad I_{j,2} \end{cases}, \quad \hat{y}_j{}^* = (y_j{}^*, b_j{}^*). \tag{1.14}$$

Then $\hat{y}_j^* \in \hat{Y}_j$. We now define

$$\hat{\Delta}^{\epsilon}_{\delta',l} = \text{the restriction } \hat{z}_{1,l} \text{ of } \hat{\Delta}^{\epsilon}_{\delta,1} \text{ to } I_{1,l} \ (l = 1, 2).$$

Let

$$\hat{z}_1 = \begin{cases} \hat{z}_{1,1} & \text{on } I_{1,1} \\ \hat{z}_{1,2} & \text{on } I_{1,2} \end{cases}. \tag{1.15}$$

Next we define

$$\hat{\Delta}^{\epsilon}_{\delta',l+2}(\hat{z}_{1,1}, \hat{y}_{1,1}, \hat{z}_{1,2}, \hat{y}_{1,2}) = \text{restriction of } \Delta^{\epsilon}_{\delta,2}(\hat{z}_1, \hat{y}_1{}^*) \quad \text{to} \quad I_{2,l} \ (l = 1, 2) \tag{1.16}$$

where $\hat{y}_1{}^*$, $\hat{z}_1$ are given by (1.14), (1.15) respectively. If $\hat{z}_{1,1}$, $\hat{z}_{1,2}$ are such that their second components are not equal (i.e., $c_{1,1} \neq c_{1,2}$) then the right-hand side of (1.16) does not make sense. But in this case it is irrelevant how one defines the left-hand side; thus we take an arbitrary definition, such as

$$\hat{\Delta}^{\epsilon}_{\delta',l+2}(\cdots) = (\bar{z}, 0)$$

where $\bar{z}$ is a constant function with values in Z. We now proceed in general. For any $\hat{z}_i$ ($1 \leqslant i \leqslant j$) in $\hat{Z}_i$, if

$$\hat{z}_i = \begin{cases} \hat{z}_{i,1} & \text{on } I_{i,1} \\ \hat{z}_{i,2} & \text{on } I_{i,2} \end{cases},$$

we define

$$\hat{\Delta}^{\epsilon}_{\delta',l+2j+2}(\hat{z}_{11}, \hat{y}_{11}, \hat{z}_{12}, \hat{y}_{12}, \ldots, \hat{z}_{j1}, \hat{y}_{j1}, \hat{z}_{j2}, \hat{y}_{j2}) = \text{restriction of}$$
$$\hat{\Delta}^{\epsilon}_{\delta,j+1}(\hat{z}_1, \hat{y}_1{}^*, \ldots, \hat{z}_j, \hat{y}_j{}^*) \text{ to } I_{j+1,l} \qquad (l = 1, 2) \tag{1.17}$$

where $\hat{y}_i{}^*$ is defined by (1.14); if $c_{i1} \neq c_{i2}$ for at least one $i \leqslant j$, we define $\hat{\Delta}^{\epsilon}_{\delta',l+2j+2}(\cdots)$ in an arbitrary way.

Now, (1.10) implies (see [11, p. 212]) that

$$V^{\delta} + \epsilon \geqslant P(\hat{y}', \hat{\Delta}_{\delta}{}^{\epsilon}(\hat{y})) \qquad \text{for any control} \quad \hat{y}', \tag{1.18}$$

where $\hat{\Delta}_{\delta}{}^{\epsilon}(\hat{y})$ is the $\hat{z}$ resulting when y is playing $\hat{y}$ and z is playing with the δ-strategy $\hat{\Delta}_{\delta}{}^{\epsilon}$. If $\hat{y}$ is any vector given by (1.12) (i.e., any outcome of a δ'-game), then the vector $\hat{y}^*$ with components as in (1.14) satisfies

$$S \leqslant S^* \leqslant S + 1/2^n \tag{1.19}$$

where S, S^* are the stopping times for $\hat{y}$ and $\hat{y}^*$ respectively. Denote by T and T^* the stopping times for $\hat{\Delta}_\delta{}^\epsilon(\hat{y})$ and $\hat{\Delta}_\delta{}^\epsilon(\hat{y}^*)$ where $\hat{y}$ is given by (1.12). It is then clear from (1.17) that $T^* = T$. Hence

$$P(\hat{y}, \hat{\Delta}_{\delta'}^\epsilon(\hat{y})) = \int_\tau^{S \wedge T} h \left[\exp \int_\tau^t k\right] dt + \chi_{S \leqslant T}\, g_1(x(S), S) \left[\exp \int_\tau^S k\right]$$
$$+ \chi_{T<S}\, g_2(x(T), T) \left[\exp \int_\tau^T k\right], \tag{1.20}$$

$$P(\hat{y}^*, \hat{\Delta}_\delta{}^\epsilon(\hat{y}^*)) = \int_\tau^{S^* \wedge T} h \left[\exp \int_\tau^t k\right] dt + \chi_{S^* \leqslant T}\, g_1(x(S^*), S^*) \left[\exp \int_\tau^{S^*} k\right]$$
$$+ \chi_{T<S^*}\, g_2(x(T), T) \left[\exp \int_\tau^T k\right]. \tag{1.21}$$

If $T < S$, then the right-hand sides of (1.20), (1.21) agree. Now, since $g_2 \geqslant g_1$, we have

$$P(\hat{y}^*, \hat{\Delta}_\delta{}^\epsilon(\hat{y}^*)) \geqslant \int_\tau^{S^* \wedge T} h \left[\exp \int_\tau^t k\right] dt + \chi_{S^* \leqslant T}\, g_1(x(S), S) \left[\exp \int_\tau^S k\right]$$
$$+ \chi_{T<S^*}\, g_1(x(T), T) \left[\exp \int_\tau^T k\right]. \tag{1.22}$$

If $S \leqslant T < S^*$, the right-hand side of (1.22) is

$$\int_\tau^T h \left[\exp \int_\tau^t k\right] dt + g_1(x(T), T) \left[\exp \int_\tau^T k\right] \geqslant P(\hat{y}, \hat{\Delta}_{\delta'}^\epsilon(\hat{y})) - C/2^n$$
$$(C \text{ constant}).$$

If $S \leqslant S^* \leqslant T$, then

$$P(\hat{y}^*, \hat{\Delta}_\delta{}^\epsilon(\hat{y}^*)) = \int_\tau^{S^*} h \left[\exp \int_\tau^t k\right] dt + g_1(x(S^*), S^*) \left[\exp \int_\tau^{S^*} k\right]$$
$$\geqslant P(\hat{y}, \hat{\Delta}_{\delta'}^\epsilon(\hat{y})) - C/2^n.$$

Hence in all cases we have

$$P(\hat{y}^*, \hat{\Delta}_\delta{}^\epsilon(\hat{y}^*)) \geqslant P(\hat{y}, \hat{\Delta}_{\delta'}^\epsilon(\hat{y})) - C/2^n. \tag{1.23}$$

Since (1.18) holds with $\hat{y}' = \hat{y}^*$, we conclude from (1.23) that

$$V^\delta + \epsilon \geqslant P(\hat{y}, \hat{\Delta}_{\delta'}^\epsilon(\hat{y})) - C/2^n.$$

Since $\hat{y}$ is an arbitrary control in a δ'-game, it follows (cf. [10, p. 212]) that

$$V^\delta + \epsilon \geqslant V^{\delta'} - C/2^n.$$

Taking $\epsilon \to 0$ and setting $U_n = V^\delta$, $U_{n+1} = V^{\delta'}$, we get

$$U_n \geqslant U_{n+1} - C/2^n,$$

but then, the sequence $\{U_n + 1/n\}$ is monotone decreasing for large n. Consequently $\lim U_n$ exists. This completes the proof of the theorem.

We shall now assume

$$\begin{aligned} f(x, t, y, z) &= f^1(x, t, y) + f^2(x, t, z), \\ h(x, t, y, z) &= h^1(x, t, y) + h^2(x, t, z). \end{aligned} \tag{1.24}$$

THEOREM 1.2. *Let* (A), (B) *and* (1.24) *hold. Then* $V^+(\xi, \tau) = V^-(\xi, \tau)$, *i.e., the value* $V(\xi, \tau)$ *exists.*

Proof. The proof is similar to the proof of Theorem 2.3.1 of [11, p. 39]. First, for any $\epsilon > 0$, there exist $\hat{\Gamma}_*{}^\delta$ and $\hat{\Delta}_*{}^\delta$ such that

$$V^\delta \leqslant P[\hat{\Delta}_\delta, \hat{\Gamma}_*{}^\delta] + \epsilon \qquad \text{for all } \hat{\Delta}_\delta, \tag{1.25}$$

$$V_\delta \geqslant P[\hat{\Gamma}_\delta, \hat{\Delta}_*{}^\delta] - \epsilon \qquad \text{for all } \hat{\Gamma}_\delta. \tag{1.26}$$

We now modify $\hat{\Delta}_*{}^\delta = (\hat{\Delta}_*^{\delta,1}, \ldots, \hat{\Delta}_*^{\delta,N})$ into $\hat{\Delta}_{**}^\delta = (\hat{\Delta}_{**}^{\delta,1}, \ldots, \hat{\Delta}_{**}^{\delta,N})$ as follows. If

$$\hat{\Delta}_*^{\delta,j}(\hat{y}_1, \hat{z}_1, \ldots, \hat{y}_{j-1}, \hat{z}_{j-1}, \hat{y}_j) = \hat{z}_j, \; \hat{z}_j = (z_j, c_j)$$

then

$$\hat{\Delta}_{**}^{\delta,j}(\hat{y}_1, \hat{z}_1, \ldots, \hat{y}_{j-1}, \hat{z}_{j-1}, \hat{y}_j) = \hat{z}_j{}^*,$$

where $\hat{z}_j{}^* = (z_j, c_j{}^*)$ and

$$c_j{}^* = \begin{cases} c_j & \text{if } b_j \equiv 0, \\ 1 & \text{if } b_j \equiv 1. \end{cases}$$

Thus if T and T^* are the stopping times corresponding to $\hat{\Delta}_*{}^\delta(\hat{y})$ and $\hat{\Delta}_{**}^\delta(\hat{y})$ respectively, and if S is the stopping time for $\hat{y}$, then

$$\begin{aligned} T < S \quad &\text{implies} \quad T^* = T < S, \\ S \leqslant T \quad &\text{implies} \quad T^* = S. \end{aligned} \tag{1.27}$$

From the form of (1.4) it is clear that

$$P(\hat{y}, \hat{\Delta}_{**}^\delta(\hat{y})) = P(\hat{y}, \hat{\Delta}_*{}^\delta(\hat{y})). \tag{1.28}$$

We now play an upper δ-game as in [10, p. 39]. Fix $\hat{z}_1 = (z_1, c_1)$ with $c_1 \equiv 0$. Next, let $\hat{y}_1 = \hat{\Gamma}_*^{\delta,1} \hat{z}_1$. Setting $\hat{z}_k^\tau(t) = \hat{z}_k(t+\delta)$ on I_{k-1}, we take

$$\hat{z}_j(t) = \hat{\Delta}_{**}^{\delta,j-1}(\hat{y}_1, \hat{z}_2^\tau, ..., \hat{y}_{j-2}, \hat{z}_{j-1}^\tau, \hat{y}_{j-1})(t-\delta) \qquad \text{for} \quad t \in I_j,$$

$$\hat{y}_j(t) = \hat{\Gamma}^{\delta,j}(\hat{z}_1, \hat{y}_1, ..., \hat{z}_{j-1}, \hat{y}_{j-1}, \hat{z}_j)(t) \qquad \text{for} \quad t \in I_j.$$

Denote the outcome by $\hat{y}^\delta(t)$, $\hat{z}_\delta(t)$ and the corresponding stopping times by S^δ, T_δ. By (1.25),

$$V^\delta \leqslant P(\hat{y}^\delta, \hat{z}_\delta) + \epsilon. \tag{1.29}$$

Next we play a lower δ-game: y will choose the components $\hat{y}_j$ at each interval I_j, and z will choose $\hat{\zeta}_1 = \hat{\Delta}_{**}^{\delta,1} \hat{y}_1$ and, in general,

$$\hat{\zeta}_j = \hat{\Delta}_{**}^{\delta,j}(\hat{y}_1, \hat{\zeta}_1, ..., \hat{y}_{j-1}, ..., \hat{\zeta}_{j-1}, \hat{y}_j).$$

Denote the outcome for z by $\hat{z}^\delta$. Then, by (1.26), (1.28)

$$V_\delta \geqslant P(\hat{y}^\delta, \hat{z}^\delta) - \epsilon. \tag{1.30}$$

As in [11, p. 40], we have

$$\hat{z}_\delta(t) = \hat{z}^\delta(t-\delta).$$

Hence the stopping time T^δ of $\hat{z}^\delta$ satisfies $T^\delta = T_\delta - \delta$. Since, by (1.27), $T_\delta \leqslant S$, we have $T^\delta < S$. Using the fact that $|T^\delta - T_\delta| = \delta$, the inequality $g_2 \geqslant g_1$, and Lemma 2.3.1 of [11], we conclude that

$$P(\hat{y}^\delta, \hat{z}^\delta) > P(\hat{y}^\delta, \hat{z}_\delta) - \eta(\delta)$$

where $\eta(\delta) \to 0$ if $\delta \to 0$. Using this in conjunction with (1.29), (1.30), we get

$$V^\delta \leqslant V_\delta + 2\epsilon + \eta(\delta).$$

Taking $\epsilon \to 0$, $\delta \to 0$, the assertion of the theorem follows.

Remark. Following Friedman [11], we can introduce the concept of a strategy $\hat{\Gamma} = \{\hat{\Gamma}_\delta\}$ for y and a strategy $\hat{\Delta} = \{\hat{\Delta}_\delta\}$ for z, and the concept of a saddle point. One can then prove that, under further assumptions on f, h, there exists a saddle point for the differential game with stopping time.

2. The Hamilton–Jacobi Variational Inequality for V^+

As in [11, p. 54], we have

$$V^\delta(\xi, \tau) = \inf_{\hat{z}_1} \sup_{\hat{y}_1} \cdots \inf_{\hat{z}_N} \sup_{\hat{y}_N} P_{\xi\tau}(\hat{y}, \hat{z}) \tag{2.1}$$

where

$$P(\hat{y}, \hat{z}) = P(\hat{y}_1, \hat{z}_1, ..., \hat{y}_N, \hat{z}_N).$$

We shall assume:

(C) $| k(x, t, y, z) - k(\bar{x}, \bar{t}, y, z)| \leqslant B_0(| x - \bar{x} | + | t - \bar{t} |)$,
$| h(x, t, y, z) - h(\bar{x}, \bar{t}, y, z)| \leqslant B_0(| x - \bar{x} | + | t - \bar{t} |)$,
$| g_i(x, t) - g_i(\bar{x}, \bar{t})| \leqslant B_0(| x - \bar{x} | + | t - \bar{t} |)$, $i = 1, 2$,
where B_0 is a constant.

THEOREM 2.1. *Let* (A), (B) *and* (C) *hold. For any* $R > 0$, $\epsilon > 0$, *there is a positive constant* $C_{R,\epsilon}$ *such that*

$$| V^+(\xi, \tau) - V^+(\bar{\xi}, \bar{\tau})| \leqslant C_{R,\epsilon}(| \xi - \bar{\xi} | + | \tau - \bar{\tau} |)$$

for all $0 \leqslant \tau, \bar{\tau} \leqslant T_0 - \epsilon$, $| \xi | \leqslant R$, $| \bar{\xi} | \leqslant R$.

Proof. Suppose first that $f(x, t, y, z)$ is uniformly Lipschitz continuous in t. As in [11, p. 53], we can form a one-to-one mapping ψ from the space of controls $\hat{y}$ in a δ-game over $[\tau, T_0]$ onto the space of controls for y in the interval $[\bar{\tau}, T_0]$; similarly for z. Denote the trajectory corresponding to $\psi\hat{y}$, $\psi\hat{z}$ by ψx (where x is the trajectory corresponding to $\hat{y}, \hat{z}$). If $S \leqslant T (S > T)$ then clearly $\psi S \leqslant \psi T$ $(\psi S > \psi T)$. Hence, the estimates of [11, pp. 53–56] can also be applied here to deduce that

$$| P_{\xi\tau}(\hat{y}, \hat{z}) - P_{\bar{\xi}\bar{\tau}}(\psi\hat{y}, \psi\hat{z})| \leqslant C_R(| \xi - \bar{\xi} | + | \tau - \bar{\tau} |).$$

Using (2.1), the assertion (2.2) follows.

If $f(x, t, y, z)$ is not Lipschitz continuous in t, we can still prove the Lipschitz continuity of $V^+(\xi, \tau)$ in ξ as before. The proof of the Lipschitz continuity in τ requires a different argument, similar to the one given in [16, Sect. 2].

Remark. If V^+ is Lipschitz continuous then, by a theorem of Rademacher (see [11]) it has a total derivative almost everywhere.

Later on we shall need the following.

COROLLARY 2.2. *Suppose* (A), (B) *hold and suppose that*

$$\begin{gathered} | f(x, t, y, z) - f(\bar{x}, t, y, z)| \leqslant L \mid x - \bar{x} \mid, \\ | h(x, t, y, z) - h(\bar{x}, t, y, z)| \leqslant L \mid x - \bar{x} \mid, \\ k(x, t, y, z) = k(t, y, z) \\ | g_i(x, t) - g_i(\bar{x}, t)| \leqslant L \mid x - \bar{x} \mid, \qquad i = 1, 2, \end{gathered} \tag{2.3}$$

then

$$| V^+(\xi, \tau) - V^+(\bar{\xi}, \tau)| \leqslant (T_0 + 1)L \mid \xi - \bar{\xi} \mid \exp\{(L + K)T_0\} \tag{2.4}$$

where

$$K = \int_0^{T_0} [\max_{y,z} k(t, y, z)]\, dt.$$

This follows by comparing the payoff $P_{\xi\tau}(\hat{y}, \hat{z})$, $P_{\bar{\xi}\tau}(\hat{y}, \hat{z})$, noting that the corresponding trajectories $x(t)$ and $\bar{x}(t)$ satisfy

$$| x(t) - \bar{x}(t)| \leqslant | \xi - \bar{\xi} | \exp[LT_0].$$

THEOREM 2.3. *Let* (A), (B) *hold. Then*

$$g_1(\xi, \tau) \leqslant V^+(\xi, \tau) \leqslant g_2(\xi, \tau). \tag{2.5}$$

Proof. Consider an upper δ-game with

$$\hat{\Delta}_\delta{}^* = (\hat{\Delta}_{\delta,1}^*, ..., \hat{\Delta}_{\delta,N}^*)$$

such that $\hat{\Delta}_{\delta,1}^* = (z_1, 1)$. Thus, $T = \tau$. Then for any $\hat{y}$,

$$P(\hat{y}, \hat{\Delta}_\delta{}^*(\hat{y})) = \begin{cases} g_1(\xi, \tau) & \text{if } \ S = \tau, \\ g_2(\xi, \tau) & \text{if } \ S > \tau. \end{cases}$$

Since $g_2 \geqslant g_1$, we get

$$\sup_{\hat{y}} P(\hat{y}, \hat{\Delta}_\delta{}^*(\hat{y})) = g_2(\xi, \tau).$$

This implies (cf. [11, p. 212]) that

$$V^\delta(\xi, \tau) \leqslant g_2(\xi, \tau).$$

The proof that $V^\delta(\xi, \tau) \geqslant g_1(\xi, \tau)$ is similar. In fact, taking

$$\hat{\Gamma}_*{}^\delta = (\hat{\Gamma}_*^{\delta,1}, ..., \hat{\Gamma}_*^{\delta,N})$$

with $\hat{\Gamma}_*^{\delta,1}$, whose range is a fixed element $(y_1(t), 1)$ (so that $S = \tau$), we get

$$P(\hat{\Gamma}_*{}^\delta(\hat{z}), \hat{z}) = g_1(\xi, \tau) \qquad \text{for all} \quad \hat{z}.$$

This yields that

$$V^\delta(\xi, \tau) \geqslant g_1(\xi, \tau).$$

THEOREM 2.4. *Let* (A), (B), (C) *hold. Then, almost everywhere,*

$$(\partial V^+/\partial\tau) + \min_{z\in Z}\max_{y\in Y}\{(\partial V^+/\partial\xi)\cdot f(\xi, t, y, z) + V^+k(\xi, t, y, z) + h(\xi, t, y, z)\} \geqslant 0 \tag{2.6}$$

if $V^+ > g_1$,

$$(\partial V^+/\partial\tau) + \min_{z\in Z}\max_{y\in Y}\{(\partial V^+/\partial\xi)\cdot f(\xi, t, y, z) + V^+k(\xi, t, y, z) + h(\xi, t, y, z)\} \leqslant 0 \tag{2.7}$$

if $V^+ < g_2$.

Finally, then

$$V^+(\xi, \tau) \to g_1(\xi, T_0) \qquad \text{if} \quad \tau \to T_0. \tag{2.8}$$

We refer to (2.6), (2.7) as the *Hamilton–Jacobi inequalities.*

Proof. A part of the proof is similar to the proof of Theorem 4.2.1 of [11], where the inequalities in (2.6), (2.7) are replaced by equalities. We let $\epsilon = k\delta$ and divide the δ-strategies $\hat{\Gamma}_\delta$, $\hat{\Delta}^\delta$ into two parts (cf. [11, p. 127]): $(\hat{\Gamma}_\delta = (\Gamma_\delta, \bar{\Gamma}_\delta)$, $\hat{\Delta}^\delta = (\Delta^\delta, \bar{\Delta}^\delta)$. We then have

$$\begin{aligned} V^\delta(\xi, \tau) &= \inf_{\Delta_\delta}\inf_{\bar{\Delta}_\delta}\sup_{\Gamma^\delta}\sup_{\bar{\Gamma}^\delta} P_{\xi\tau}[\Delta_\delta, \bar{\Delta}_\delta, \Gamma^\delta, \bar{\Gamma}^\delta] \\ &= \sup_{\Gamma^\delta}\inf_{\Delta_\delta}\inf_{\bar{\Delta}_\delta}\sup_{\bar{\Gamma}^\delta} P_{\xi\tau}[\Delta_\delta, \bar{\Delta}_\delta, \Gamma^\delta, \bar{\Gamma}^\delta]. \end{aligned} \tag{2.9}$$

Consider all the δ-strategies

$$\Delta_\delta^0 = (\Delta_0^{\delta,1}, \ldots, \Delta_0^{\delta,k})$$

such that their images $(\hat{z}_1, \ldots, \hat{z}_k)$ are always such that $c_1 \equiv 0, \ldots, c_k \equiv 0$ (where $\hat{z}_i(t) = (z_i(t), c_i(t))$), i.e., $T \geqslant \tau + \epsilon$. Then, by (2.9),

$$V^\delta(\xi, \tau) \leqslant \sup_{\Gamma^\delta}\inf_{\Delta_\delta^0}\inf_{\Delta_\delta}\sup_{\bar{\Gamma}^\delta} P[\Delta_\delta^0, \bar{\Delta}_\delta, \Gamma^\delta, \bar{\Gamma}^\delta]. \tag{2.10}$$

Consider $P[\Delta_\delta^0, \bar{\Delta}_\delta, \Gamma^\delta, \bar{\Gamma}^\delta]$. Suppose Γ^δ is such that, for its image $\hat{y}$, one of the b_i is $\equiv 1$, i.e., $S < \tau + \epsilon$. Since $T \geqslant \tau + \epsilon$, $S \leqslant T$. Hence

$$P[\Delta_\delta^0, \bar{\Delta}_\delta, \Gamma^\delta, \bar{\Gamma}^\delta] = g_1(x(S), S)\exp\left[\int_\tau^S k\,dt\right] + \int_\tau^S h(x, t, y, z)\exp\left[\int_\tau^t k\,ds\right]dt. \tag{2.11}$$

Now, if $V^+(\xi, \tau) > g_1(\xi, \tau)$ then

$$V^\delta(\xi, \tau) > g_1(\xi, \tau) + \alpha$$

for some α and all δ sufficiently small. Hence, if ϵ is sufficiently small, the right-hand side of (2.11) is $\leqslant g_1(\xi, \tau) + \alpha/2 \leqslant V^\delta(\xi, \tau) - \alpha/2$. Therefore

$$\inf_{\bar{\Delta}_\delta^0} \sup_{\bar{\Delta}_\delta} \sup_{\Gamma^\delta} P[\bar{\Delta}_\delta^0, \bar{\Delta}_\delta, \Gamma^\delta, \bar{\Gamma}^\delta] \leqslant V^\delta(\xi, \tau) - \alpha/2.$$

Comparing this with (2.10) it is clear that the $\sup_{\Gamma^\delta}$ can be reached only when $\bar{\Gamma}^\delta$ is such that stopping for y occurs at some time $S \geqslant \tau + \epsilon$. Therefore

$$\begin{aligned} V^\delta \leqslant \sup_{\bar{\Gamma}^\delta} \inf_{\bar{\Delta}_\delta^0} \inf_{\Delta_\delta} \sup_{\Gamma^\delta} P[\cdots] &= \sup_{\bar{\Gamma}_0^\delta} \inf_{\bar{\Delta}_\delta^0} \inf_{\Delta_\delta} \sup_{\Gamma^\delta} P[\cdots] \\ &\leqslant \inf_{\bar{\Delta}_\delta^0} \sup_{\bar{\Gamma}_0^\delta} \inf_{\Delta_\delta} \sup_{\Gamma^\delta} P[\cdots] \end{aligned} \tag{2.12}$$

where $\bar{\Gamma}_0^\delta$ runs over the δ-strategies $\bar{\Gamma}^\delta$ for which stopping occurs at some time $\geqslant \tau + \epsilon$.

Now, for a given pair $\bar{\Delta}_\delta^0, \bar{\Gamma}_0^\delta$

$$\begin{aligned} \inf_{\Delta_\delta} \sup_{\Gamma^\delta} P[\cdots] &= \inf_{\Delta_\delta} \sup_{\Gamma^\delta} \left\{ \int_\tau^{\tau+\epsilon} h \left[\exp \int_\tau^t k\right] dt + \int_{\tau+\epsilon}^{S\wedge T} h \left[\exp \int_\tau^t k\right] dt \right. \\ &\qquad \left. + \chi_{S\leqslant T} g_1 \left[\exp \int_\tau^{S\wedge T} k\, dt\right] + \chi_{T<S}\, g_2 \left[\exp \int_\tau^{S\wedge T} k dt\right] \right\} \\ &= V^\delta(\tau + \epsilon, x(\tau + \epsilon)) \left[\exp \int_\tau^{\tau+\epsilon} k\, dt\right] + \int_\tau^{\tau+\epsilon} h \left[\exp \int_\tau^t k\right] dt. \end{aligned} \tag{2.13}$$

To every δ-strategy $\bar{\Delta}_\delta^0$ there corresponds a δ-strategy Δ_δ^* in the sense of Friedman [11] (i.e., without stopping), as follows.

$$\bar{\Delta}_{\delta,j}^*(z_1, y_1, \ldots, z_{j-1}, y_{j-1}) = z_j$$

if

$$\bar{\Delta}_{\delta,j}^0(\hat{z}_1, \hat{y}_1, \ldots, \hat{z}_{j-1}, \hat{y}_{j-1}) = (z_j, 0)$$

where $\hat{z}_i = (z_i, 0)$, $\hat{y}_i = (y_i, 0)$. This mapping is one-to-one and onto the space of all lower δ-strategies of z on the open interval $(\tau, \tau + \epsilon)$. Similarly there is a correspondence $\bar{\Gamma}_0^\delta \to \Gamma_*^\delta$.

It is clear that

$$\inf_{\Delta_\delta^0}\sup_{\Gamma_0^\delta} Q[\cdots] = \inf_{\Delta_\delta^*}\sup_{\Gamma_*^\delta} Q[\cdots]$$

for any bounded payoff Q. Hence from (2.12), (2.13) we get

$$V^\delta(\xi,\tau) \leqslant \inf_{\Delta_\delta^*}\sup_{\Gamma_*^\delta} \left\{\int_\tau^{\tau+\epsilon} h\left[\exp\int_\tau^t k\right] dt \right.$$
$$\left. + V^\delta(x(\tau+\epsilon),\tau+\epsilon)\left[\exp\int_\tau^{\tau+\epsilon} k\, dt\right]\right\} \qquad (2.14)$$

where Δ_δ^*, Γ_*^δ now vary on the sets of δ-strategies, for z and y respectively (defined in the sense of Friedman [11], i.e., without stopping times), over $(\tau, \tau+\epsilon)$.

The relation (2.14) is similar to the relation (4.2.6) in [11], except that instead of equality we now have ineqaulity. The proof of the derivation of the Hamilton–Jacobi equation from (4.2.6) as given in [11], pp. 128–130], can also be applied in the present context to yield the Hamilton–Jacobi inequality (2.6), provided (ξ,τ) is a point where V^+ has a total derivative.

The proof of (2.7) is similar. Finally, the proof of (2.8) is obvious.

3. Existence of a Solution for First-Order Nonlinear Variational Inequality

We shall now consider the variational inequality

$$\begin{aligned}
&(\partial u/\partial t) + F(x,t,u,u_x) \geqslant 0 \qquad \text{if} \quad u > g_1\ (0 \leqslant t \leqslant T_0,\ x \in R^m)\\
&(\partial u/\partial t) + F(x,t,u,u_x) \leqslant 0 \qquad \text{if} \quad u < g_2\ (0 \leqslant t \leqslant T_0,\ x \in R^m)\\
&u(x,T_0) = g(x,T_0).
\end{aligned} \qquad (3.1)$$

We assume:

(F) $F(x,t,u,p)$ is continuous in $(x,t,u,p) \in R^m \times [0, T_0] \times R \times R^m$ and satisfies a Lipschitz condition in (x,u,p) and, almost everywhere,

$$\begin{aligned}
&|F(x,t,0,0)| \leqslant B_1,\\
&|F_x(x,t,u,p)| \leqslant K_1(1+|u|+|p|),\\
&|F_u(x,t,u,p)| \leqslant B_1,\\
&|F_p(x,t,u,p)| \leqslant B_1,
\end{aligned} \qquad (3.2)$$

where B_1, K_1 are positive constants.

(G) $g_1(x, t)$, $g_2(x, t)$ are continuous in $(x, t \in R^m \times [0, T_0]$ and

$$g_1(x, T_0) = g_2(x, T_0) \qquad (x \in R^m) \tag{3.3}$$

$$|g_i(x, t) - g_i(\bar{x}, t)| \leqslant L_* |x - \bar{x}|, \qquad \text{for} \quad i = 1, 2, \tag{3.4}$$

where L_* is a positive constant.

Following Fleming [6], we shall write $F(x, t, u, p)$ as an upper Hamiltonian function of a differential game, for $|u| \leqslant M$, $|p| \leqslant \mu$, and suitable M, μ. Thus, we shall exhibit functions f, h, k and control sets Y, Z such that

$$F(x, t, u, p) = \min_{z \in Z} \max_{y \in Y} \{f(x, t, y, z) \cdot p + k(x, t, y, z)u + h(x, t, y, z)\}. \tag{3.5}$$

We take

$$\begin{aligned} y &= (y', y''), |y'| \leqslant B_1, |y''| \leqslant B_2 \qquad (y' \in R, y'' \in R^m), \\ z &= (z', z''), |z'| \leqslant M, |z''| \leqslant \mu \qquad (z' \in R, z'' \in R^m); \end{aligned} \tag{3.6}$$

the numbers B_2, M, μ will be determined later on. Next,

$$\begin{aligned} f(x, t, y, z) &= F(x, t, z', z'')/(1 + |z''|^2) + y'' \\ k(x, t, y, z) &= y', \\ h(x, t, y, z) &= F(x, t, z', z'') \cdot z''/(1 + |z''|^2) - y'z' - y'' \cdot z''. \end{aligned} \tag{3.7}$$

Let $K = K(K_1, M)$ be a constant such that

$$K_1(1 + M + |z''|)/(1 + |z''|^2) \leqslant K.$$

By (3.2)

$$|f_x| \leqslant K, \qquad |h_x| \leqslant K. \tag{3.8}$$

LEMMA 3.1. *Given any positive number M, there is a positive number $B_2 = B_2(B_1, M)$ depending on B_1, M, such that for any positive constant μ,* (3.5) *holds.*

A similar lemma with the order of $\min_z \max_y$ interchanged was proved by Fleming [6]. The present lemma can be proved by the same arguments.

Now consider the differential game with stopping time associated with f, k, h, g_1, g_2. Denote by $V^+(x, t)$ its upper value, when the initial condition is $x(t) = x$. We shall prove

$$|V^+(x, t)| \leqslant N, \tag{3.9}$$

$$|V_x^+(x, t)| \leqslant N^* \qquad \text{a.e.} \tag{3.10}$$

where $N = \max(\sup |g_1|, \sup |g_2|)$ and $N^* = N^*(B_1, K_1, M)$ is a positive constant.

To prove (3.9) let $\hat{\Gamma}^\delta$ be a δ-strategy for y for which the stopping time is $S = \tau$ (when the initial condition is $x(\tau) = \xi$). Then

$$|P(\hat{\Gamma}^\delta(\hat{z}), \hat{z})| \leqslant N.$$

It follows that $V^\delta \geqslant -N$, hence $V^+ \geqslant -N$. Similarly, let $\hat{\Delta}_\delta$ be a δ-strategy for z for which $T = \tau$. Then

$$|P(\hat{y}, \hat{\Delta}_\delta(\hat{y}))| \leqslant N.$$

This yields $V^\delta \leqslant N$, hence $V^+ \leqslant N$.

As for the estimate (3.10), this follows from Corollary 3.2 and the inequalities in (3.8).

We now choose $M = N + 1$, $B_2 = B_2(B_1, N + 1)$ and $\mu = N^*(B_1, K_1, N + 1)$. By Lemma 3.1, for almost all (x, t), the equality (3.5) holds for $u = V^+, p = V_x{}^+$. Applying Theorem 2.4 we then obtain the following result.

THEOREM 3.2. *Let* (F), (G) *hold. Then there exists a solution of the variational inequality* (3.1), *namely* $u = V^+$, *where* V^+ *is the upper value of the differential game with stopping time corresponding to* f, k, h, g_1, g_2 *and* Y, Z *given by* (3.6), (3.7).

Remark. Suppose one of the constraints is missing; for instance, g_2 and T do not appear and $\chi_{S \leqslant T}$ is replaced by 1. All the results of the previous sections also extend to this case. The only result which requires a new argument is Theorem 3.2. Thus, the proof of the inequality

$$V^\delta \leqslant N \tag{3.11}$$

is no longer valid when z does not use stopping times. We now proceed to define

$$\hat{\Delta}^\delta \quad \text{by} \quad \hat{\Delta}^\delta(\hat{y}) \equiv 0. \qquad \text{Then} \quad h \equiv 0.$$

Consequently, if we assume, in addition to (3.2), that

$$|F(x, t, u, 0)| \leqslant B_1, \tag{3.12}$$

then we get

$$P(\hat{y}, \hat{\Delta}^\delta(\hat{y})) \leqslant N,$$

where N is a positive constant depending only on B_1. This yields (3.11).

4. Nonlinear Parabolic Variational Inequalities

Let $\epsilon > 0$. We consider the following problem.

$$
\begin{gathered}
(\partial u/\partial t) + (\epsilon^2/2)\,\Delta u + F(x, t, u, u_x) \geqslant f \quad \text{if} \quad u > 0 \; (0 < t < T_0\,,\, x \in R^m),\\
(\partial u/\partial t) + (\epsilon^2/2)\,\Delta u + F(x, t, u, u_x) \leqslant f \quad \text{if} \quad u < g \; (0 < t < T_0\,,\, x \in R^m),\\
u(x, T_0) = g(x, T_0) \qquad (x \in R^m),\\
0 \leqslant u(x, t) \leqslant g(x, t) \qquad (0 \leqslant t \leqslant T_0\,,\, x \in R^m).
\end{gathered}
\tag{4.1}
$$

In the case $F(x, t, u, p)$ is linear in u and in p the existence, uniqueness and regularity of the solution were established by Bensoussan–Lions [1] and Friedman [14, 15]. In the present section we shall employ the notations and techniques of Friedman [15]. Thus we work with norms

$$
|\, u\,|_{i,p,\mu}^{G} = \Big\{ \sum_{|\alpha| \leqslant i} \int_G e^{-\mu p |x|}\, |\, D_x^{\alpha} u\,|^p \, dx \Big\}^{1/p},
$$

$$
|\, u\,|_{i,p,\mu} = |\, u\,|_{i,p,\mu}^{R^m}
$$

and denote the corresponding spaces by $W^{i,p,\mu}(G)$, $W^{i,p,\mu}$. Here μ is any nonnegative number, and $p \geqslant 2$.

The spaces $W_0^{i,p,\mu}(G)$ are the completion in $W^{i,p,\mu}(G)$ of the C^∞ functions with compact support in G.

We shall need the following assumptions.

(F_0) $F(x, t, u, p)$ is continuous in $(x, t, u, p) \in R^m \times [0, T_0] \times R \times R^m$, Lipschitz continuous in (x, u, p) and, almost everywhere,

$$
\begin{aligned}
&|F(x, t, u, p)| \leqslant C(1 + |\, u\,| + |\, p\,|),\\
&|F_t(x, t, u, p)| \leqslant C(1 + |\, u\,| + |\, p\,|),\\
&|F_u(x, t, u, p)| \leqslant C,\\
&|F_p(x, t, u, p)| \leqslant C,
\end{aligned}
$$

where C is a constant.

(G_0) (i) $g(x, t)$ is measurable and $\geqslant 0$ for $(x, t) \in R^m \times [0, T_0]$;

(ii) $g, g_t, g_{x_i}, g_{x_i x_j}$ belong to $L^p(0, T_0\,; W^{0,p,\mu}) \cap L^2(0, T_0\,; W^{0,2,\mu})$;

(iii) $g_{tt} \in L^2(0, T_0\,; W^{0,2,\mu})$;

(iv) $g(x, T_0) \in W^{2-2/p,p,\mu} \cap W^{1,2,\mu}$;

(v) $f \in L^p(0, T_0; W^{0,p,\mu}) \cap L^2(0, T_0; W^{0,2,\mu})$,
$f_t \in L^2(0, T_0; W^{0,2,\mu})$,

where p is a positive number $\geqslant 2$.

Notice that (ii) implies that

$$g \in C([0, T_0]; W^{0,p,\mu} \cap W^{0,2,\mu}).$$

Also, (v) implies that

$$f \in C([0, T_0]; W^{0,2,\mu}).$$

Observe finally that in (iv) we are using $W^{l,p,\mu}$ where l is not necessarily an integer. The definition of $W^{l,p}$ is standard, and the definition of $W^{l,p,\mu}$ is obtained from the definition of $W^{l,p}$ in the obvious way.

THEOREM 4.1. *If* (F_0), (G_0) *hold then there exists a unique solution* u *of* (4.1) *in the following sense*:

$$u \in L^\infty(0, T_0; W^{1,2,\mu}) \cap L^p(0, T_0; W^{2,p,\mu}) \cap L^2(0, T_0; W^{2,2,\mu}),$$

$$\partial u/\partial t \in L^p(0, T_0; W^{0,p,\mu}) \cap L^2(0, T_0; W^{0,2,\mu}), \tag{4.2}$$

$$0 \leqslant u \leqslant g \quad a.e., \tag{4.3}$$

$$((\partial u/\partial t) + (\epsilon^2/2)\,\Delta u + F(x, t, u, u_x) - f)\,(v - u) \leqslant 0 \quad a.e.\ for\ all\ v,$$
$$0 \leqslant v \leqslant g \quad a.e., \tag{4.4}$$

$$u(x, T_0) = g(x, T_0) \quad a.e. \tag{4.5}$$

Proof. Let $\zeta_R(x)$ be a nonnegative C^∞ function such that $\zeta_R = 1$ if $|x| < R - 1$, $\zeta_R = 0$ if $|x| \geqslant R$, $|D_x^\alpha \zeta_R| \leqslant C_\alpha$ $(0 \leqslant |\alpha| \leqslant 2)$ where C_α is a constant independent of R. Set $g_R = \zeta_R g$, $\hat{g}_R = \zeta_R \hat{g}$.

Consider the initial-boundary value problem:

$$\frac{\partial u}{\partial t} + \frac{\epsilon^2}{2}\Delta u - \frac{1}{\beta}(u - g_R)^+ + \frac{1}{\beta}u^- + F(x, t, u, u_x) = f \tag{4.6}$$
$$\text{a.e.}\quad (x \in \Omega_R,\ 0 < t < T_0),$$

$$u(t) \in W_0^{1,2}(\Omega_R) \quad \text{a.e.}, \tag{4.7}$$

$$u(T_0) = g_R(T_0), \tag{4.8}$$

where $g_R = g_R(t) = g_R(\cdot, t)$ and $\Omega_R = \{x;\ |x| < R\}$. By either Lions [17] or Friedman [9] and Fleming [5] one can show the existence of a unique solution of (4.6)–(4,8) for any $R > 1$, $\beta > 0$.

We next establish several inequalities.

Let us multiply (4.6) by $-\rho^p \mid u \mid^{p-2} u$ where $\rho = \exp(-\mu \mid x \mid)$. We get, after integrating over Ω_R and using the first inequality in (F_0),

$$-\frac{1}{p}\frac{d}{dt}\int_{\Omega_R} \rho^p \mid u \mid^p dx - \frac{\epsilon^2}{2}\int_{\Omega_R} \rho^p \mid u \mid^{p-2} u\Delta u dx$$
$$\leqslant \int_{\Omega_R} \rho^p \left[\mid u \mid^{p-1}(\mid f \mid + C) + C \mid u \mid^p + C \mid u \mid^{p-1} \left|\frac{\partial u}{\partial x}\right|\right] dx.$$

Since

$$-\int_{\Omega_R} \rho^p \mid u \mid^{p-2} u\Delta u dx$$
$$= \int_{\Omega_R} \left[(p-1)\rho^p \mid u \mid^{p-2} \left|\frac{\partial u}{\partial x}\right|^2 - p\mu\rho^p \mid u \mid^{p-2} u \sum_i \frac{x_i}{|x|}\frac{\partial u}{\partial x_i}\right] dx,$$

we get, for any $\gamma > 0$,

$$-\frac{1}{p}\frac{d}{dt}\int_{\Omega_R} \rho^p \mid u \mid^p dx + \int_{\Omega_R} \rho^p \mid u \mid^{p-2} \left|\frac{\partial u}{\partial x}\right|^2 \left((p-1)\frac{\epsilon^2}{2} - \frac{\epsilon^2}{4}p\mu\gamma - \frac{C\gamma}{2}\right) dx$$
$$\leqslant \int_{\Omega_R} \rho^p(\mid f \mid + C)\mid u \mid^{p-1} dx + \int_{\Omega_R} \mid \rho \mid^p \mid u \mid^p \left(C + \frac{C}{2\gamma} + \frac{p\mu\epsilon^2}{4\gamma}\right) dx$$
$$\leqslant \frac{1}{p}\int_{\Omega_R} \rho^p(\mid f \mid + C)^p dx + \int_{\Omega_R} \rho^p \mid u \mid^p \left(\frac{p-1}{p} + C + \frac{C}{2\gamma} + \frac{p\mu\epsilon^2}{4\gamma}\right) dx. \tag{4.9}$$

Choosing γ so that

$$(p-1)(\epsilon^2/2) - (\epsilon^2/4)\, p\mu\gamma - C\gamma/2 > 0$$

and setting

$$\phi(t) = \int_{\Omega_R} \rho^p \mid u(x, t)\mid^p dx,$$

we get from (4.9) that

$$-\phi'(t) \leqslant \delta\phi(t) + C_1$$

where δ, C_1 are positive constants. It follows that $\phi(t) \leqslant$ const, i.e.,

$$\int_{\Omega_R} \rho^p \mid u(x, t)\mid^p dx \leqslant C. \tag{4.10}$$

Since the above analysis also applies for $p = 2$, we get

$$\int_{\Omega_R} \rho^2 \mid u(x, t)\mid^2 dx \leqslant C. \tag{4.11}$$

Taking $p = 2$ in (4.9) and integrating with respect to t, we get, after using (4.11),

$$\int_0^{T_0} \int_{\Omega_R} \rho^2 \left| \frac{\partial u}{\partial x} \right|^2 dxdt \leqslant C$$

for some positive constant C. Together with (4.11) this yields

$$\int_0^{T_0} \left(| u(t)|_{1,2,\mu}^{\Omega_R} \right)^2 dt \leqslant C. \tag{4.12}$$

Next we multiply (4.6) by $-\rho^p[(u - g_R)^+]^{p-1}$ and integrate with respect to x, t. Using the relations

$$\begin{aligned}
&\int_0^{T_0} \int_{\Omega_R} \rho^p \frac{\partial u}{\partial t} [(u - g_R)^+]^{p-1} \, dxdt \\
&= \int_0^{T_0} \int_{\Omega_R} \rho^p \frac{\partial (u - g_R)^+}{\partial t} [(u - g_R)^+]^{p-1} \, dxdt \\
&\quad + \int_0^{T_0} \int_{\Omega_R} \rho^p \frac{\partial g_R}{\partial t} [(u - g_R)^+]^{p-1} \, dxdt \\
&= -\frac{1}{p} \int_{\Omega_R} \rho^p [(u(x, t) - g_R(x, t))^+]^p \, dx \\
&\quad + \int_0^{T_0} \int_{\Omega_R} \rho^p \frac{\partial g_R}{\partial t} [(u - g_R)^+]^{p-1} \, dxdt,
\end{aligned}$$

$$\begin{aligned}
&\int_0^{T_0} \int_{\Omega_R} \rho^p \Delta u \cdot [(u - g_R)^+]^{p-1} \, dxdt \\
&= \int_0^{T_0} \int_{\Omega_R} \rho^p \Delta (u - g_R) \cdot [(u - g_R)^+]^{p-1} \, dxdt \\
&\quad + \int_0^{T_0} \int_{\Omega_R} \rho^p \Delta g_R \cdot [(u - g_R)^+]^{p-1} \, dxdt \\
&= -\int_0^{T_0} \int_{\Omega_R} (p - 1) \rho^p \, [(u - g_R)^+]^{p-2} \left| \frac{\partial}{\partial x} (u - g_R)^+ \right|^2 dxdt \\
&\quad + p\mu \int_0^{T_0} \int_{\Omega_R} \rho^p [(u - g_R)^+]^{p-1} \sum \frac{x_i}{|x|} \frac{\partial}{\partial x_i} (u - g_R)^+ \, dxdt \\
&\quad + \int_0^{T_0} \int_{\Omega_R} \rho^p \Delta g_R \cdot [(u - g_R)^+]^{p-1} \, dxdt,
\end{aligned}$$

$$\left| \int_0^{T_0} \int_{\Omega_R} \rho^p F(x, t, u, u_x) \, [(u - g_R)^+]^{p-1} \, dxdt \right|$$
$$\leqslant C \left[\int_0^{T_0} \int_{\Omega_R} \rho^p (1 + |u|^p + |u_x|^p) \, dxdt \right]^{1/p}$$
$$\times \left[\int_0^{T_0} \int_{\Omega_R} \rho^p [(u - g_R)^+]^p \, dxdt \right]^{(p-1)/p},$$

$$\int_0^{T_0} \int_{\Omega_R} \rho^p k [(u - g_R)^+]^{p-1} \, dxdt$$
$$\leqslant \left[\int_0^{T_0} \int_{\Omega_R} \rho^p |k|^p \, dxdt \right]^{1/p}$$
$$\times \left[\int_0^{T_0} \int_{\Omega_R} \rho^p [(u - g_R)^+]^p \, dxdt \right]^{(p-1)/p}$$

for $k = f$ and $k = \Delta g_R + \partial g_R/\partial t$, and noting that

$$\int_0^{T_0} \int_{\Omega_R} \rho^p [(u - g_R)^+]^{p-1} \frac{x_i}{|x|} \frac{\partial}{\partial x_i} (u - g_R)^+ \, dxdt$$
$$\leqslant \frac{1}{\gamma} \left[\int_0^{T_0} \int_{\Omega_R} \rho^p [(u - g_R)^+]^p \, dxdt \right]^{(p-1)/p}$$
$$+ \frac{\gamma}{4} \int_0^{T_0} \int_{\Omega_R} \rho^p \left| \frac{\partial}{\partial x_i} (u - g_R)^+ \right|^2 [(u - g_R)^+]^{p-2}$$

we arrive at the inequality

$$\frac{1}{\beta} \int_0^{T_0} \int_{\Omega_R} \rho^p [(u - g_R)^+]^p \, dxdt$$
$$\leqslant C \left(1 + \int_0^{T_0} \int_{\Omega_R} \rho^p \left| \frac{\partial u}{\partial x} \right|^p dxdt \right)^{1/p} \left[\int_0^{T_0} \int_{\Omega_R} \rho^p [(u - g_R)^+]^p \, dxdt \right]^{(p-1)/p};$$

here we have used (G_0) (ii), (v). It follows that

$$\int_0^{T_0} \left(\left| \frac{1}{\beta} (u - g_R)^+ \right|_{0,p,\mu} \right)^p dt \leqslant C + C \int_0^{T_0} \int_{\Omega_R} \rho^p \left| \frac{\partial u}{\partial x} \right|^p dxdt. \quad (4.13)$$

Similarly, if we multiply (4.6) by $-\rho^p(u^-)^{p-1}$ and integrate, we get

$$\int_0^{T_0} \left(\left|\frac{1}{\beta} u^-\right|_{0,p,\mu}\right)^p dt \leqslant C + C \int_0^{T_0} \int_{\Omega_R} \rho^p \left|\frac{\partial u}{\partial x}\right|^p dxdt. \tag{4.14}$$

From (4.6) and (4.13), (4.14) it follows that

$$\int_0^{T_0} \int_{\Omega_R} \rho^p \left|\frac{\partial u}{\partial t} + \frac{\epsilon^2}{2} \Delta u\right|^p dxdt \leqslant C + C \int_0^{T_0} \int_{\Omega_R} \rho^p \left|\frac{\partial u}{\partial x}\right|^p dxdt. \tag{4.15}$$

Friedman [15] has extended the standard L^p elliptic estimates to the case of increasing domains Ω_R, with constants independent of R. The idea is to use a special partition of unity and then apply the estimates in each small domain.

We shall now do something similar for the L^p parabolic estimate of Solonnikov [18], in the special case of the operator $(\partial u/\partial t) + (\epsilon^2/2) \Delta u$. For a fix domain Ω_R and for a function u vanishing for $x \in \partial\Omega_R$, $0 < t < T_0$, the estimate asserts that

$$\begin{aligned} &\int_0^{T_0} \int_{\Omega_R} \rho^p[| u |^p + | u_t |^p + | u_x |^p + | u_{xx} |^p] \, dxdt \\ &\qquad \leqslant C \int_0^{T_0} \int_{\Omega_R} \rho^p \left[\left| u_t + \frac{\epsilon^2}{2} \Delta u \right|^p \eta(| u |^p + | u_x |^p)\right] dxdt \\ &\qquad\quad + C \, | g_R(\, , T_0)|^{\Omega_R}_{2-2/p,p,\mu} \end{aligned} \tag{4.16}$$

where $\rho \equiv 1$, $\eta = 0$.

Now we take a partition of unity of Ω_R, say $\{\phi_i\}$, as in [15], and apply (4.16) (with $\rho = 1$) to $u\phi_i \exp(-\mu \mid x_i \mid)$, where x_i is a point in the support of ϕ_i. The constant C can now be taken to be independent of i, R. Summing the resulting inequalities over i, we end up with the inequality (4.16) (for $\rho = \exp(-\mu \mid x \mid)$) with constants C and η that are independent of R.

Evaluating the right-hand side of (4.16) by (4.15), and using the inequality (see, e.g. [10])

$$\int_{\Omega_R} | u_x |^p \, dx \leqslant \lambda \int_{\Omega_R} | u_{xx} |^p \, dx + \frac{C}{\lambda} \int_{\Omega_R} | u |^p \, dx$$

which holds for any $\lambda > 0$ with a constant C independent of λ, R, we obtain, after recalling (4.10),

$$\int_0^{T_0} \int_{\Omega_R} \rho^p(| u |^p + | u_t |^p + | u_x |^p + | u_{xx} |^p) \, dxdt \leqslant C \tag{4.17}$$

where C is a constant independent of β, R.

If we use (4.17) in (4.13), (4.14), we get

$$\int_0^{T_0} \left(\left| \frac{1}{\beta} (u - g_R)^+ \right|_{0,p,\mu} \right)^p dt \leqslant C, \tag{4.18}$$

$$\int_0^{T_0} \left(\left| \frac{1}{\beta} u^- \right|_{0,p,\mu} \right)^p dxdt \leqslant C. \tag{4.19}$$

We need one more estimate. Differentiate (4.6) with respect to t, multiply by $\rho^2(\partial u/\partial t)$ and integrate with respect to (x, t). Using the last two inequalities in (F_0), we get

$$\begin{aligned}\int_{\Omega_R} \rho^2 \mid u_t(x, t)|^2 \, dx - \int_{\Omega_R} \rho^2 \mid u_t(x, T_0)|^2 \, dx + \frac{\epsilon^2}{2} \int_t^{T_0} \int_{\Omega_R} \rho^2 \mid u_{xt} \mid^2 dxdt \\ + \int_t^{T_0} \int_{\Omega_R} \frac{\rho^2}{\beta} \frac{\partial (u - g_R)^+}{\partial t} \frac{\partial u}{\partial t} \, dxdt - \int_t^{T_0} \int_{\Omega_R} \frac{\rho^2}{\beta} \frac{\partial u^-}{\partial t} \frac{\partial u}{\partial t} \, dxdt = G \end{aligned} \tag{4.20}$$

where

$$| G | \leqslant C \int_t^{T_0} \int_{\Omega_R} \rho^2 [| u_t |(1 + | u | + | u_t | + | u_x | + | u_{xt} |) + | f_t |^2] \, dxdt.$$

Now,

$$\begin{aligned}\int_t^{T_0} \int_{\Omega_R} \frac{\rho^2}{\beta} \frac{\partial (u - g_R)^+}{\partial t} \frac{\partial u}{\partial t} \, dxdt &= \int_t^{T_0} \int_{\Omega_R} \frac{\rho^2}{\beta} \frac{\partial (u - g_R)^+}{\partial t} \frac{\partial (u - g_R)^+}{\partial t} \, dxdt \\ &+ \int_t^{T_0} \int_{\Omega_R} \frac{\rho^2}{\beta} \frac{\partial (u - g_R)^+}{\partial t} \frac{\partial g_R}{\partial t} \, dxdt \geqslant \int_t^{T_0} \int_{\Omega_R} \frac{\rho^2}{\beta} \frac{\partial (u - g_R)^+}{\partial t} \frac{\partial g_R}{\partial t} \, dxdt \\ &= - \int_{\Omega_R} \frac{\rho^2}{\beta} (u - g_R)^+ \frac{\partial g_R}{\partial t} \, dx - \int_t^{T_0} \int_{\Omega_R} \frac{\rho^2}{\beta} (u - g_R)^+ \frac{\partial^2 g_R}{\partial t^2} \, dxdt.\end{aligned}$$

Similarly,

$$- \int_t^{T_0} \int_{\Omega_R} \frac{\rho^2}{\beta} \frac{\partial u^-}{\partial t} \frac{\partial u}{\partial t} \, dxdt \geqslant - \int_t^{T_0} \int_{\Omega_R} \frac{\rho^2}{\beta} (u^-)^2 \, dxdt.$$

Since $u_t(x, T_0) = f(x, T_0) - [(\epsilon^2/2) \Delta g_R - F(x, T_0, g_R, (\partial/\partial x) g_R)]_{t=T_0}$, we also have

$$\int_{\Omega_R} \rho^2 \mid u_t (x, T_0)|^2 \, dx \leqslant C.$$

Using these relations in (4.20), and using (4.18), (4.19), we get

$$\int_{\Omega_R} [\rho^2 \mid u_t \mid^2]_{t=s}\, dx + \frac{\epsilon^2}{2} \int_s^{T_0} \int_{\Omega_R} \rho^2 \mid u_{xt} \mid^2 dxdt$$
$$+ \int_{\Omega_R} \left[\frac{\rho^2}{\beta} (u - g_R)^+ \frac{\partial g_R}{\partial t}\right]_{t=s} dx \leqslant C.$$

Integrating with respect to s, and using (4.18), we get

$$\int_0^{T_0} \int_{\Omega_R} \rho^2 \mid u_t \mid^2 dxdt + \frac{\epsilon^2}{2} \int_0^{T_0} \left[\int_s^{T_0} \int_{\Omega_R} \rho^2 \mid u_{xt} \mid^2 dxdt\right] ds \leqslant C. \quad (4.21)$$

Now, for any $\eta > 0$, all the functions given in Theorem 4.1 can be extended to $-\eta \leqslant t \leqslant 0$ such that all the conditions in (G_0), (F_0) remain valid in the interval $[-\eta, T_0]$ instead of $[0, T_0]$. We can therefore carry out all the previous analysis in the interval $[-\eta, T_0]$. In particular, (4.21) yields

$$\int_{-\eta}^{T_0} \left[\int_s^{T_0} \int_{\Omega_R} \rho^2 \mid u_{xt} \mid^2 dxdt\right] ds \leqslant C.$$

Hence

$$\int_{-\eta}^{0} \left[\int_0^{T_0} \int_{\Omega_R} \rho^2 \mid u_{xt} \mid^2 dxdt\right] ds \leqslant C,$$

i.e.,

$$\int_0^{T_0} \int_{\Omega_R} \rho^2 \mid u_{xt} \mid^2 dxdt \leqslant C, \quad (4.22)$$

with a different constant C.

Denote the solution of (4.6)–(4.8) by u_R. Extend u_R to $\tilde{u}_R$ defined on $0 \leqslant t \leqslant T_0$, $x \in R^m$, such that $\tilde{u}_R = 0$ if $\mid x \mid > R + 1$, and

$$\int_0^{T_0} \int_{R^m} \rho^2 \left[\mid \tilde{u}_R \mid^q + \left|\frac{\partial}{\partial t} \tilde{u}_R\right|^q + \left|\frac{\partial}{\partial x} \tilde{u}_R\right|^q + \left|\frac{\partial^2}{\partial x^2} \tilde{u}_R\right|^q\right] dxdt \leqslant C$$
$$(q = p, 2),$$

$$\int_0^{T_0} \int_{R^m} \rho^2 \left|\frac{\partial^2}{\partial t \partial x} \tilde{u}_R\right|^2 dxdt \leqslant C, \qquad \mid \tilde{u}_R(t)\mid_{1,2,\mu} \leqslant C.$$

Using Sobolev's compact imbedding theorem (see, e.g. [10]), we obtain a sequence $\{\tilde{u}_{R_n}\}$ which is strongly convergent and a.e. convergent in compact subsets of $R^m \times [0, T_0]$ to a function u, together with its first x-derivative. Hence

$$F(x, t, \tilde{u}_{R_n}, (\partial/\partial x)\tilde{u}_{R_n}) \to F(x, t, u, \partial u/\partial x) \quad \text{a.e.} \quad \text{in } (x, t).$$

We may further assume that

$$\tilde{u}_{R_n} \to u \text{ in } L^p(0, T_0 ; W^{2,p,\mu}) \cap L^2(0, T_0 ; W^{2,2,\mu}) \quad \text{weakly},$$

$$\tilde{u}_{R_n} \to u \text{ in } L^\infty(0, T_0 ; W^{1,2,\mu}) \quad \text{weak star},$$

$$(\partial/\partial t)\tilde{u}_{R_n} \to (\partial u/\partial t) \text{ in } L^p(0, T_0 ; W^{0,p,\mu}) \cap L^2(0, T_0 ; W^{0,2,\mu}) \quad \text{weakly}.$$

Taking $R = R_n \to \infty$ in (4.6), we find that

$$(\partial u/\partial t) + \frac{\epsilon^2}{2} \Delta u - \frac{1}{\beta}(u - g)^+ + \frac{1}{\beta} u^- + F(x, t, u, u_x) = f \quad \text{a.e.} \tag{4.23}$$

Further,

$$\int_0^{T_0} (| u |_{2,p,\mu})^p \, dt \leqslant C, \qquad \int_0^{T_0} (| u |_{2,2,\mu})^2 \, dt \leqslant C, \qquad | u(t)|_{1,2,\mu} \leqslant C,$$

$$\int_0^{T_0} \left(\left| \frac{\partial u}{\partial t} \right|_{0,p,\mu} \right)^p dt \leqslant C, \qquad \int_0^{T_0} \left(\left| \frac{\partial u}{\partial t} \right|_{0,2,\mu} \right)^2 dt \leqslant C,$$

$$u(T_0) = g(T_0).$$

Denote this function u by u_β. From (4.18), (4.19) we have

$$\int_0^{T_0} \left(\frac{1}{\beta} |(u_\beta(t) - g(t))^+ |_{0,2,\mu} \right)^2 dt \leqslant C, \tag{4.24}$$

$$\int_0^{T_0} \left(\frac{1}{\beta} | u_\beta^-(t)|_{0,2,\mu} \right)^2 dt \leqslant C. \tag{4.25}$$

We now extract a subsequence $\beta = \beta_n \to 0$ such that

$$u_\beta \to u \text{ in } L^p(0, T_0 ; W^{2,p,\mu}) \cap L^2(0, T_0 ; W^{2,2,\mu}) \quad \text{weakly},$$

$$u_\beta \to u \text{ in } L^\infty(0, T_0 ; W^{1,2}) \quad \text{weak star},$$

$$\partial u_\beta/\partial t \to \partial u/\partial t \text{ in } L^p(0, T_0 ; W^{0,p,\mu}) \cap L^2(0, T_0 ; W^{0,2,\mu}) \quad \text{weakly},$$

$$u_\beta \to u \text{ in } L^2(0, T_0 ; W^{1,2}(G)) \quad \text{strongly for any bounded domain } G,$$

$$u_\beta \to u \text{ a.e. in } (x, t,)$$

$$\partial u_\beta/\partial x \to \partial u/\partial x \quad \text{a.e.} \quad \text{in } (x, t).$$

As in Bensoussan–Lions [1] it follows from (4.24), (4.25) that

$$[u(t) - g(t)]^+ = 0, \qquad u^-(t) = 0.$$

Now let $0 \leqslant v \leqslant g$. Multiplying (4.23) by $v - u_\beta$ and noticing that

$$[(1/\beta)u_\beta^- - (1/\beta)\,(u_\beta - g)^+]\,(v - u_\beta) \geqslant 0$$

we get an inequality. If we multiply this inequality by a nonnegative function $\phi(t)$ in $C_0^\infty(0, T_0)$, and integrate with respect to $(x, t) \in G \times (0, T_0)$, we get, after taking $\beta = \beta_n \to 0$,

$$\int_0^{T_0} \left\{\int_G \left[\frac{\partial u}{\partial t} + \frac{\epsilon^2}{2}\,\Delta u + F(x, t, u, u_x) - f\right](v - u)\,dx\right\} \phi(t)\,dt \geqslant 0,$$

where G is any bounded domain. This implies (4.4). It is also clear that (4.3), (4.5) and (4.2) hold.

To prove uniqueness, we suppose that u_1, u_2 are two solutions of (4.2)–(4.5). We write (4.4) first for $u = u_1$, $v = u_2$ and then for $u = u_2$, $v = u_1$, and take the sum. Employing the uniform Lipschitz continuity of $F(x, t, u, p)$ in u, p, we easily obtain (cf. [1] and [15])

$$-(d/dt)\,(|\, u_1(t) - u_2(t)|_{0,2,\mu})^2 \leqslant \beta(|\, u_1(t) - u_2(t)|_{0,2,\mu})^2,$$

where β is a positive constant. Since $u_1(T_0) - u_2(T_0) = 0$, we conclude that $u_1 = u_2$ a.e.

We shall state a generalization of Theorem 4.1. First we need some assumptions.

(D) (i) $a_{ij}(x, t)$, $\partial a_{ij}(x, t)/\partial x_l$, $b_i(x, t)$, $c(x, t)$ and their first t-derivatives are measurable and bounded functions in $R^m \times [0, T_0]$.

(ii) For all $0 \leqslant t \leqslant T_0$, $x \in R^m$, $\xi \in R^m$,

$$\sum_{i,j=1}^m a_{ij}(x, t)\,\xi_i \xi_j \geqslant \alpha\,|\,\xi\,|^2 \qquad (\alpha > 0).$$

Set

$$Lu \equiv \sum_{i,j=1}^m a_{ij}\,\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^m b_i\,\frac{\partial u}{\partial x_i} + cu.$$

(G) (i) The function g_1 is a measurable function in $(x, t) \in R^m \times [0, T_0]$ and

$$g_1 \in L^p(0, T_0\,;\, W^{2,p,\mu}) \cap L^2(0, T_0\,;\, W^{2,2,\mu}) \cap L^\infty(0, T_0\,;\, W^{1,2,\mu}),$$
$$\partial g_1/\partial t \in L^p(0, T_0\,;\, W^{0,p,\mu}) \cap L^2(0, T_0\,;\, W^{0,2,\mu});$$

(ii) the function

$$\tilde{f}(t) = f(t) - (\partial g_1/\partial t) - Lg_1 - F(x, t, g_1, \partial g_1/\partial x)$$

satisfies:

$$\tilde{f} \in L^p(0, T_0; W^{0,p,\mu}) \cap L^2(0, T_0; W^{0,2,\mu}), \qquad \partial\tilde{f}/\partial t \in L^2(0, T_0; W^{0,2,\mu});$$

(iii) the function $g \equiv g_2 - g_1$ satisfies the conditions in (G_0)(i)–(iv).

THEOREM 4.2. *If the conditions* (D), (F_0), (G) *hold, then there exists a unique solution of the nonlinear parabolic variational inequality*:

$$u \in L^p(0, T_0; W^{2,p,\mu}) \cap L^2(0, T_0; W^{2,2,\mu}) \cap L^\infty(0, T_0; W^{1,2}),$$
$$\partial u/\partial t \in L^p(0, T_0; W^{0,p,\mu}) \cap L^2(0, T_0; W^{0,2,\mu}), \tag{4.27}$$

$$g_1 \leqslant u \leqslant g_2 \quad a.e., \tag{4.28}$$

$$((\partial u/\partial t) + Lu + F(x, t, u, u_x) - f)(v - u) \leqslant 0 \quad a.e. \text{ for all } v,$$
$$g_1 \leqslant v \leqslant g_2 \quad a.e., \tag{4.29}$$

$$u(x, T_0) = g_1(x, T_0) \quad a.e. \tag{4.30}$$

To prove the theorem, we first notice that Theorem 4.1 holds when $(\epsilon^2/2)\,\Delta u$ is replaced by Lu, provided (D) holds. The proof, in fact, is similar to the proof of Theorem 4.1, except for some changes (as in [15]) arising from the fact that the a_{ij} are not constants. Once Theorem 4.1 is known for L, Theorem 4.2 follows by applying Theorem 4.1 with $g = g_2 - g_1$ and f replaced by $\tilde{f}$.

Remark 1. In [15] a result similar to Theorem 4.1 was proved for F linear in u, u_x provided $\partial g(x, t)/\partial t \geqslant 0$. However there is an error in [15] in the assertion that $[\partial(u - g)^+/\partial t]\,[\partial u/\partial t] \geqslant 0$. This last inequality holds, in general, only if $\partial g/\partial t = 0$, i.e., $g = g(x)$. Thus, this assumption must be incorporated into [15]. Notice though that the solution u established in [15] is in $L^\infty(0, T_0; W^{2,p,\mu} \cap W^{2,2,\mu})$. If, in particular, $p > n$ then, for almost all t, $u_x(x, t)$ is Hölder continuous in $x \in R^m$. The method of Friedman [15] extends to the case where $F(x, t, u, u_x)$ is a nonlinear function of u, u_x satisfying (F_0). Thus, if $g_2(x, t) - g_1(x, t) = g(x)$, and if

$$g \in W^{2,p,\mu} \cap W^{2,2,\mu},$$
$$g_1 \in L^\infty(0, T_0; W^{2,p,\mu} \cap W^{2,2,\mu}),$$
$$\partial g_1/\partial t, \tilde{f} \quad \text{and} \quad \partial\tilde{f}/\partial t \quad \text{belong to} \quad L^\infty(0, T_0; W^{0,p,\mu} \cap W^{0,2,\mu}),$$

then the solution u of Theorem 4.2 satisfies:

$$u \in L^\infty(0, T_0; W^{2,p,\mu} \cap W^{2,2,\mu}),$$
$$\partial u/\partial t \in L^\infty(0, T_0; W^{0,p,\mu} \cap W^{0,2,\mu}).$$

Remark 2. Brezis [3] considered linear parabolic variational inequalities with time dependent constraints (i.e., with $g = g(x, t)$) in a cylinder with bounded base. Our estimates, when specialized to $p = 2$ and to a cylinder with a bounded base, coincide with his estimates.

5. Stochastic Differential Games with Stopping Times

We consider a stochastic differential system described by

$$dx = f(x, t, y, z)\,dt + \sigma(x, t)\,dw, \tag{5.1}$$
$$x(\tau) = \xi, \tag{5.2}$$

where $w(t)$ is m-dimensional Brownian motion, $\tau \in [0, T_0)$, $\xi \in R^m$, f is m vector and $\sigma = (\sigma_{ij})$ is $m \times m$ matrix. We are going to introduce a concept of differential game with Markov stopping times S, T associated with (5.1), (5.2) and a payoff

$$\begin{aligned} P(y, S; z, T) = P_{\xi\tau} \Big\{ &\int_\tau^{S\wedge T} \Big[\exp \int_\tau^t k(x, s, y, z)\,ds\Big] h(x, t, y, z)\,dt \\ &+ \Big[\exp \int_\tau^S k(x, t, y, z)\,dt\Big] g_1(x(S), S)\chi_{S\leqslant T} \\ &+ \Big[\exp \int_\tau^T k(x, t, y, z)\,dt\Big] g_2(x(T), T)\chi_{T<S} \end{aligned}$$

where χ_A is the indicator function of a set A, and k, h, g_1, g_2 are given functions.

The model will be a generalization of that introduced by Friedman [12] in case $S \equiv T \equiv T_0$.

Setting

$$a_{ij} = \sum_k \sigma_{ik}\sigma_{kj},$$

we shall need the conditions:

$$\sum a_{ij}(x, t)\,\xi_i\xi_j \geqslant \alpha\,|\,\xi\,|^2, \qquad \alpha > 0, \tag{5.4}$$

$$a_{ij},\ \partial a_{ij}/\partial x_l,\ \partial^2 a_{ij}/\partial x_l \partial t \text{ are continuous and bounded in } (x, t) \in R^m \times [0, T], \tag{5.5}$$

$$f,\ \partial f/\partial x_i,\ \partial f/\partial y,\ \partial f/\partial z \text{ are continuous and bounded in } (x, t) \in R^m \times [0, T_0], \tag{5.6}$$

$$h, k, f \text{ are continuous functions with their } (x, t) \text{ first derivatives, and} \tag{5.7}$$

$$|h| + |h_t| + |k| + |k_t| + |f| + |f_t| + |f_x| \leqslant C,$$

$$|g_1| + |g_2| \leqslant C(1 + |x|^\beta), \text{ for some } \beta > 0.$$

Let Y, Z be compact sets in some euclidean spaces.

A control function for y is a measurable function $y(x, t)$ with values in Y. Similarly, a control function for z is a measurable function $z(x, t)$ with values in Z.

For any pair of control functions $y = y(x, t)$, $z = z(x, t)$ there is a unique solution x of (5.1), (5.2) in the sense of Stroock–Varadhan [19] (under the assumptions that (5.4) holds, (a_{ij}) is continuous and f is continuous and bounded).

On the other hand if we restrict ourselves to control functions which are Lipschitz continuous in x, then there exists a unique solution of (5.1), (5.2) in the sense of Ito (provided we also assume that $f(x, t, y, z)$ is Lipschitz continuous in x, y, z).

Remark. For simplicity we shall usually assume that (5.6) holds. Then we can take the solution of (5.1), (5.2) in either sense. However all the results in this section and in the following ones remain valid when the condition (5.6) is omitted, provided the solution of (5.1), (5.2) is taken in the Stroock–Varadhan sense.

The stopping times S, T are taken with respect to the σ-algebra $\mathscr{F}_t$ generated by $x(s)$, $\tau \leqslant s \leqslant t$. We assume that

$$\tau \leqslant S,\ T \leqslant T_0 .$$

The player y tries to choose $y(x, t)$, S so as to maximize the payoff, and the player z tries to choose $z(x, t)$, T so as to minimize the payoff.

The system made up of (5.1), (5.2), (5.3) and Y, Z will be referred to as a *stochastic differential game with stopping time.*

DEFINITION. A pair $\{(y^*(x, t), S^*),\ (z^*(x, t), T^*)\}$ is called a *saddle point* if

$$P(y, S; z^*, T^*) \leqslant P(y^*, S^*; z^*, T^*) \leqslant P(y^*, S^*; z, T) \tag{5.8}$$

for all controls y, z and stopping times S, T.

We shall now assume the *minimax condition* (cf. [12]).

$$\begin{aligned}
&\max_{y\in Y}\min_{z\in Z}\{h(x,t,y,z)+p\cdot f(x,t,y,z)+uk(x,t,y,z)\}\\
&\quad=\min_{z\in Z}\max_{y\in Y}\{h(x,t,y,z)+p\cdot f(x,t,y,z)+uk(x,t,y,z)\}\\
&\quad\equiv H(x,t,u,p). \qquad (5.9)
\end{aligned}$$

Define

$$Lu=\frac{1}{2}\sum_{i,j=1}^{m}a_{ij}\frac{\partial^2 u}{\partial x_i\partial x_j}. \qquad (5.10)$$

Notice that (5.7) implies the condition (F_0) for the function $F = H$. We shall now assume:

the functions g_1, g_2 satisfy all the conditions in (G), with $F = H$ and L given by (5.10). (5.11)

Then we can apply Theorem 4.2 to conclude that there exists a unique solution u of the variational inequality:

$$\begin{aligned}
&u\in L^p(0,T_0;W^{2,p,\mu})\cap L^2(0,T_0;W^{2,2,\mu})\cap L^\infty(0,T_0;W^{1,2}),\\
&\partial u/\partial t\in L^p(0,T_0;W^{0,p,\mu})\cap L^2(0,T_0;W^{0,2,\mu}),
\end{aligned} \qquad (5.12)$$

$$((\partial u/\partial t)+Lu+H(x,t,u,u_x))\,(v-u)\leqslant 0 \quad \text{a.e.} \quad \text{for every } v,\ g_1\leqslant v\leqslant g_2 \quad \text{a.e.} \qquad (5.13)$$

$$g_1\leqslant u\leqslant g_2 \quad \text{a.e.} \quad \text{in } (x,t), \qquad (5.14)$$

$$u(T_0)=g_1(T_0). \qquad (5.15)$$

We shall now assume that

$$p>m. \qquad (5.16)$$

By Sobolev's inequality it then follows that the functions g_1, g_2 and u are continuous in $(x, t) \in R^m \times [0, T_0]$. Notice that under the conditions of Remark 1 at the end of Section 4, if $p > m$ then, for almost all t, $u_x(x, t)$ satisfies a Hölder condition in x with coefficient and exponent which do not depend on t.

Let $y^*(x, t)$ and $z^*(x, t)$ be any measurable functions which render $\max_y$ and $\min_z$ in (5.9) when $u = u(x, t)$, $p = u_x(x, t)$.

Define sets

$$\begin{aligned}
C_1&=\{(x,t);\,u(x,t)>g_1(x,t)\},\\
C_2&=\{(x,t);\,u(x,t)<g_2(x,t)\}.
\end{aligned} \qquad (5.17)$$

Denote by S^* and T^* the first hitting times of the sets $R^m \backslash C_1$ and $R^m \backslash C_2$ respectively. We shall show that $\{(y^*, S^*), (z^*, T^*)\}$ is a saddle point.

Before we do that we wish to point out that if $y^*(x, t)$, $z^*(x, t)$ are only known to be measurable functions (and not Lipschitz in x) then we must take the meaning of (5.1), (5.2) in the sense of Stroock-Varadhan [19]. If however they are Lipschitz in x, we can take (5.1), (5.2) in the sense of Ito, and restrict all control functions to be Lipschitz continuous in x.

THEOREM 5.1. *Let the conditions* (5.4)–(5.7), (5.9), (5.11), (5.16) *hold. Then* $\{(y^*, S^*), (z^*, T^*)\}$ *is a saddle point of the stochastic differential game with stopping time* (5.1)–(5.3).

Proof. We shall prove the second inequality in (5.8). Let $z(x, t)$ be a control function for z and T a stopping time for the process $\tilde{x}$ determined by (5.1), (5.2) when $z = z(x, t)$, $y = y^*(x, t)$. Denote by x^* the process determined by (5.1), (5.2) when $z = z^*(x, t)$, $y = y^*(x, t)$. If we apply Ito's formula to the function

$$u(x, t) \left[\exp \int_\tau^t k(x, s, y^*(x, s), z^*(x, s))\, ds\right]$$

and the process $\tilde{x}$, formally, then we get

$$\begin{aligned} E_{\xi\tau} &\left\{u(\tilde{x}(S^* \wedge T), S^* \wedge T) \left[\exp \int_\tau^{S^* \wedge T} k\,dt\right]\right\} \\ &= u(\xi, \tau) + E_{\xi\tau} \left\{\int_\tau^{S^* \wedge T} \left\{\left[\exp \int_\tau^t k\,ds\right] [(\partial u/\partial t) + u_x \cdot f(x, t, y^*, z)\right.\right. \\ &\qquad \left.\left. + uk(x, t, y^*, z) + Lu[(\tilde{x}(t), t)\right\} dt\right\}. \end{aligned} \tag{5.18}$$

Suppose for the moment that (5.18) can be rigorously justified. Notice that

$$\text{if} \quad t < S^* \wedge T \leqslant S^* \qquad \text{then} \qquad u(\tilde{x}(t), t) > g_1(\tilde{x}(t), t);$$

hence

$$[(\partial u/\partial t) + Lu + H(x, t, u, u_x)]\, (\tilde{x}(t), t) \geqslant 0.$$

From the definition of y^* we then have

$$\begin{aligned} [(\partial u/\partial t) + Lu + h(x, t, y^*(x, t), z) + u_x \cdot f(x, t, y^*(x, t), z) \\ + uk(x, t, y^*(x, t), z)]\, (\tilde{x}(t), t) \geqslant 0 \end{aligned} \tag{5.19}$$

for every $z \in Z$, with equality when $z = z^*(x, t)$.

Taking $z = z(\tilde{x}(t), t)$ in (5.19) and using the resulting inequality in (5.18), we get

$$E_{\xi\tau}\left\{u(\tilde{x}(S^* \wedge T), S^* \wedge T)\left[\exp \int_\tau^{S^*\wedge T} k dt\right] \geqslant u(\xi, \tau)\right.$$
$$\left. - E_{\xi\tau}\left\{\int_\tau^{S^*\wedge T}\left\{\left[\exp \int_\tau^t k ds\right] h(x, t, y^*(x, t), z^*(x, t))\right\}(\tilde{x}(t), t))\, dt\right\}\right.$$

with equality when $z = z^*(x, t)$.

Noticing that

$$u(\tilde{x}(S^* \wedge T), S^* \wedge T) \leqslant \chi_{S^* \leqslant T}\, g_1(\tilde{x}(S^*), S^*) + \chi_{T<S^*}\, g_2(\tilde{x}(T), T)$$

with equality when $z = z^*(x, t)$, $T = T^*$, the second inequality in (5.8) follows.

In order to justify (5.18) rigorously we introduce the fundamental solution $q(\tau, \xi, t, y)$ of L, and interpret the integral

$$E_{\xi\tau} \int_\tau^{S^*\wedge T} \phi(\tilde{x}(t), t)\, dt$$

on the right-hand side of (5.18) as

$$\int_{R^m} \int_\tau^{T_0} \phi(y, t)\, q(\tau, \xi, t, y)\, E(\chi_{S^*\wedge T} \mid \tilde{x}(t) = y)\, dy dt.$$

With this interpretation (5.18) can be justified as in Bensoussan–Lions [1]. We can then proceed, as before, to verify the second inequality in (5.8).

The proof of the first inequality in (5.8) is similar.

Remark. Suppose we restrict the Markov times S, T to be hitting times of domains of the form

$$G = \{(x, t) |\, x \in B_t\, ,\ \tau \leqslant t \leqslant T_0\}$$

with smooth lateral boundary $\partial_0 G$, then we can interpret (5.8) as follows. Let G_{S^*} be the domain whose hitting time is S^*. Let G_T be any domain whose hitting time is T. Let v be a solution of

$$(\partial v/\partial t) + Lv + (\partial v/\partial x) \cdot f(x, t, y^*(x, t), z(x, t)) + k(x, t, y^*(x, t), z(x, t))v$$
$$= h(x, t, y^*(x, t), z(x, t)) \text{ in } G_{T\wedge S^*} \equiv G_T \cap G_{S^*}\, ,$$
$$v = g_1 \quad \text{on} \quad (\partial_0 G_{T\wedge S^*}) \cap G_{S^*}\, ,$$
$$v = g_2 \quad \text{on} \quad (\partial_0 G_{T\wedge S^*}) \cap G_T\, ,$$
$$v = g_1 \quad \text{on} \quad t = T_0\, .$$

Then

$$P_{\xi\tau}(y^*, S^*; z, t) = v(\xi, \tau).$$

Letting $v = v^*$ when $T = T^*$, the second inequality in (5.8) means that

$$v^*(\xi, \tau) \leqslant v(\xi, \tau). \tag{5.20}$$

Thus (z^*, T^*) is a solution of a certain deterministic problem in which the controls are both some coefficients of the parabolic equation and the domain where the parabolic initial boundary value problem is solved.

6. Convergence of u_ϵ to V in Case of Control Theory

We specialize the results of Section 5 to the case where f, h, k do not depend on y and g_1 does not appear. Thus, we are dealing with the case of one player z, who tries to minimize the payoff.

We define

$$H(x, t, u, p) = \min_{z \in Z}\{h(x, t, y, z) + p \cdot f(x, t, y, z) + uk(x, t, y, z)\} \tag{6.1}$$

and assume:

The functions f, k, h, g_2 satisfy all the conditions in (5.6), (5.7), and (G_0) holds for $g = g_2$, $F = H$. (6.2)

Set $g = g_2$. Then, by Theorem 4.2 (slightly modified) there exists a unique solution u_ϵ of the variational inequality:

$$\begin{aligned}
&u_\epsilon \in L^p(0, T_0; W^{2,p,\mu}) \cap L^2(0, T_0; W^{2,2,\mu}) \cap L^\infty(0, T_0; W^{1,2,\mu}),\\
&\partial u_\epsilon/\partial t \in L^p(0, T_0; W^{0,p,\mu}) \cap L^2(0, T_0; W^{0,2,\mu}),\\
&((\partial u_\epsilon/\partial t) + (\epsilon^2/2)\, \Delta u_\epsilon + H(x, t, u_\epsilon, \partial u_\epsilon/\partial x))\, (v - u_\epsilon) \leqslant 0,\\
&\qquad\qquad \text{a.e.} \quad \text{for any } v,\ v \leqslant g \quad \text{a.e.},\\
&u_\epsilon \leqslant g \quad \text{a.e.}\\
&u_\epsilon(T_0) = g(T_0).
\end{aligned} \tag{6.3}$$

We now consider the first order variational inequality

$$\begin{aligned}
&((\partial V/\partial t) + H(x, t, V, \partial V/\partial x))\, (v - V) \leqslant 0 \quad \text{a.e. for all } v;\\
&\qquad\qquad v \leqslant g \quad \text{a.e.},\\
&V \leqslant g \quad \text{a.e.},\\
&V(T_0) = g(T_0)\\
&V \text{ locally Lipschitz continuous.}
\end{aligned} \tag{6.4}$$

It follows from Theorems 2.3, 2.4 (specialized to the case where g_1 and S do not appear and $\chi_{T<S}$ is replaced by 1) that V^+ is a solution of (6.4). We shall write $V = V^+$.

Let $(\Omega, \mathscr{F}, P)$ be a fixed probability space and let $\mathscr{F}_t$ be an increasing family of sub σ-algebras of $\mathscr{F}$. Following Fleming and Rishel [8] we define *admissible nonanticipative control and stopping time* as a pair $(z(t), T)$ where z is a stochastic process and T a random time defined on $(\Omega, \mathscr{F}, P)$ and having the following properties.

$$z(t) \in Z, \qquad \text{for} \quad t \in [\tau, T_0], \tau \leqslant T \leqslant T_0 ; \tag{6.5}$$

$z(t)$ is nonanticipative with respect to $\mathscr{F}_t$ and T is a stopping time with respect to $\mathscr{F}_t$; (6.6)

given $\xi \in R^m$, there exists a Brownian motion $b(t)$ such that the stochastic differential equation (6.7)

$$dx_\epsilon = f(x_\epsilon, t, z)\, dt + \epsilon db$$
$$x_\epsilon(\tau) = \xi$$

has a solution which is unique in probability.

[If we take this equation in the sense of Stroock–Varadhan [19], we may choose $\Omega = C(\tau, T_0 ; R^m)$, $\mathscr{F}_t$ the σ-algebra generated by ω_s, $s \leqslant t$, and z, T any process and stopping time satisfying (6.5), (6.6)]. We will denote $\mathscr{U}_\tau$ by the class of admissible nonanticipative controls and stopping times.

We now define, for $(z(\cdot), T) \in \mathscr{U}_\tau$,

$$P^\epsilon_{\xi\tau}(z(\cdot), T) = E_{\xi\tau} \left\{ \int_\tau^T \left[\exp \int_\tau^t k(x_\epsilon(s), s, z)\, ds \right] h(x_\epsilon(t), t, z)\, dt \right.$$
$$\left. + g(x_\epsilon(T), T) \left[\exp \int_\tau^T k(x_\epsilon(t), t, z)\, dt \right] \right\}. \tag{6.8}$$

We have

LEMMA 6.1. *Let* (6.2) *hold. Then*

$$u_\epsilon(\xi, \tau) = \operatorname*{Inf}_{(z(\cdot), T) \in \mathscr{U}_\tau} P^\epsilon_{\xi\tau}(z(\cdot), T). \tag{6.9}$$

Proof. Apply Ito's formula to $u_\epsilon(\xi, \tau)$ and the process $x_\epsilon(t)$ as in the proof of (5.8).

LEMMA. 6.2. *Let* (6.2) *hold. Then*

$$V(\xi, \tau) = \operatorname*{Inf}_{(z(\cdot), T) \in \mathscr{U}_\tau} P^0_{\xi\tau}(z(\cdot), T). \tag{6.10}$$

Proof. We know already that

$$V(\xi, \tau) = \operatorname*{Inf}_{(z(\cdot), T \in \tilde{\mathscr{U}}_\tau)} P^0_{\xi\tau}(z(\cdot), T) \tag{6.11}$$

where $\tilde{\mathscr{U}}_\tau$ is the class of *deterministic* measurable functions from (τ, T_0) into Z, and $T \in [\tau, T_0]$. Since clearly $\tilde{\mathscr{U}}_\tau \subset \mathscr{U}_\tau$, we have

$$V(\xi, \tau) \geqslant \operatorname*{Inf}_{(z(\cdot), T) \in \mathscr{U}_\tau} P^0_{\xi\tau}(z(\cdot), T). \tag{6.12}$$

Now for ω fixed, $z(t, \omega)$ and $T(\omega)$ belong to $\tilde{\mathscr{U}}_\tau$. Therefore

$$\int_\tau^{T(\omega)} \left[\exp \int_\tau^t k(x(s, \omega), s, z(\omega))\, ds\right] h(x, t, z(\omega))\, dt$$
$$+ g(x(T(\omega)), T(\omega)) \left[\exp \int_\tau^{T(\omega)} k(x, t, z(\omega))\, dt\right] \geqslant V(\xi, \tau),$$

and by taking the mathematical expectation, we get

$$P^0_{\xi\tau}(z(\cdot), T) \geqslant V(\xi, \tau), \quad \text{for every} \quad (z(\cdot), T) \in \mathscr{U}_\tau$$

which with (6.12) implies (6.11).

THEOREM 6.1. *Let* (6.2) *hold. Then, for each* $(\xi, \tau) \in R^m \times [0, T_0]$,

$$u_\epsilon(\xi, \tau) \to V(\xi, \tau) \qquad as \quad \epsilon \to 0. \tag{6.13}$$

Proof. As easily verified, for any $(z(\cdot), T) \in \mathscr{U}_\tau$ we have

$$| P^\epsilon_{\xi\tau}(z(\cdot), T) - P^0_{\xi\tau}(z(\cdot), T)| \leqslant C(\epsilon)$$

where $C(\epsilon) \to 0$, if $\epsilon \to 0$; $C(\epsilon)$ is independent of $(z(\cdot), T)$. But this and Lemmas 6.1, 6.2 imply that

$$| u_\epsilon(\xi, \tau) - V(\xi, \tau)| \leqslant C(\epsilon),$$

which proves the theorem.

Remark. When there is no stopping time, Theorem 6.3 is just the classical result of stochastic control theory, due to Fleming [7]. Fleming also states conditions on a function $F(x, t, u, p)$ in order that it has the form of a Hamiltonian of a control problem. When this is the case, Theorem 6.1 shows that the solution u_ϵ of the nonlinear parabolic variational inequality (6.3) (with $H = F$) has a limit V, as $\epsilon \to 0$, and the limit is a solution of the first order nonlinear variational inequality (6.4).

7. Convergence of u_ϵ to V^+ in the General Case

In Section 1 we let both players choose stopping times. In this section we let only one of the players choose a stopping time. For definiteness, let y choose a stopping time. Denote by $V^+(\xi, \tau)$ the corresponding upper value. We shall prove that $V^+(\xi, \tau)$ is the limit of solutions $u_\epsilon(\xi, \tau)$ of nonlinear parabolic variational inequalities as in Theorem 4.2 with $F = H^+, L = (1/2)\, \epsilon^2 \Delta, g_2 = \infty$. Using Theorem 3.2 (or, rather, the remark following it), we can look upon this result also in a different manner, namely:

The solution u_ϵ in Theorem 4.2, with $L = (1/2)\, \epsilon^2 \Delta$, $g_2 \equiv \infty$ has a limit as $\epsilon \to 0$, the limit V^+ being a particular solution of the limiting first order nonlinear variational inequality. Further, V^+ is the upper value of a certain differential game with stopping time (for y only) whose dynamics and payoff can be expressed explicitly in terms of F.

Let f, k, h, g_1 be given as in Section 1. Set

$$H^+(x, t, u, p) = \min_{z \in Z} \max_{y \in Y} \{p \cdot f(x, t, y, z) + uk(x, t, y, z) + h(x, t, y, z)\}.$$

We shall assume:

$$f, k, h, g_1 \quad \text{are continuously differentiable and}$$
$$\begin{aligned} &|f|, |f_x|, |f_t|, |k|, |k_x|, |k_t|, |h|, |h_x|, |h_t|, |g_1|, \\ &\qquad\qquad\qquad\qquad |\partial g_1/\partial x|, |\partial g_1/\partial t| \end{aligned} \tag{7.2}$$
are bounded.

These conditions imply, of course, the conditions analogous to (A), (B), (C) of Section 1 with g_2 missing. Consequently, $V^+(x, t)$ is Lipschitz continuous in (x, t), uniformly in compact subsets of $[0, T_0) \times R^m$, and, a.e.,

$$(\partial V^+/\partial t) + H^+(x, t, V^+, (\partial/\partial x)V^+) = 0 \qquad \text{if} \quad V^+ > g_1, \tag{7.3}$$

$$(\partial V^+/\partial t) + H^+(x, t, V^+, (\partial/\partial x)V^+) \leqslant 0 \qquad \text{if} \quad V^+ \geqslant g_1, \tag{7.4}$$

$$V^+(x, T_0) = g_1(x, T_0). \tag{7.5}$$

The condition (7.2) implies the condition corresponding to (F_0) for $F = H^+$ and g_2 missing.

Next we assume:

$$
\begin{aligned}
& g_1 \in L^p(0, T_0 ; W^{2,p,\mu}) \cap L^2(0, T_0 ; W^{2,2,\mu}) \\
& \partial g_1/\partial t \in L^p(0, T_0 ; W^{0,p,\mu}) \cap L^2(0, T_0 ; W^{0,2,\mu}) \\
& \partial^2 g_1/\partial t^2 \in L^2(0, T_0 ; W^{0,2,\mu}), \\
& g_1 \in L^\infty(0, T_0 ; W^{1,2,\mu}), \\
& \Delta g_1 \text{ and } F(\cdot, \cdot, g_1, (\partial/\partial x) g_1) \text{ belong to } L^p(0, T_0 ; W^{0,p,\mu}) \cap L^2(0, T_0 ; W^{0,2,\mu}) \\
& \qquad \text{and their first } t\text{-derivatives belong to } L^2(0, T_0 ; W^{0,2,\mu}), \\
& p > m.
\end{aligned}
\tag{7.6}
$$

Here μ is any positive number. Notice that the condition (G) in Section 4 (with g_2 missing) follows from (7.6). Therefore, when (7.2), (7.6) hold, there exists (by the proof of Theorem 4.2) a unique solution of the nonlinear parabolic variational inequality:

$$
\begin{aligned}
& u_\epsilon \in L^p(0, T_0 ; W^{2,p,\mu}) \cap L^2(0, T_0 ; W^{2,2,\mu}) \cap L^\infty(0, T_0 ; W^{1,2,\mu}), \\
& \partial u_\epsilon/\partial t \in L^p(0, T_0 ; W^{0,p,\mu}) \cap L^2(0, T_0 ; W^{0,2,\mu}), \\
& u_\epsilon \geqslant g_1 \quad \text{a.e.}, \\
& [(\partial u_\epsilon/\partial t) + (\epsilon^2/2)\, \Delta u_\epsilon + H^+(x, t, u_\epsilon, \partial u_\epsilon/\partial x)]\, (v - u_\epsilon) \leqslant 0 \quad \text{a.e.} \\
& \qquad\qquad \text{for any } v, v \geqslant g_1 \quad \text{a.e.}, \\
& u_\epsilon(x, T_0) = g_1(x, T_0) \quad \text{a.e.}
\end{aligned}
\tag{7.7}
$$

Notice that since $p > m$, $u(x, t)$ is continuous in $(x, t) \in R^m \times [0, T_0]$.

Let R^* be a positive number such that all the solutions of (1.1), (1.2) satisfy $|x(t)| \leqslant R^*$ if $\tau \leqslant t \leqslant T_0$. We shall need the following condition:

$$
h + \partial g_1/\partial t + \partial g_1/\partial x \cdot f + k g_1 \leqslant 0 \qquad \text{if} \quad |x| \leqslant R^*, \tau \leqslant t \leqslant T_0 . \tag{7.8}
$$

The main result of this section is the following theorem.

THEOREM 7.1. *Let* (7.2), (7.6) *and* (7.8) *hold. Then, for any* $(\xi, \tau) \in R^m \times [0, T_0]$,

$$
u_\epsilon(\xi, \tau) \to V^+(\xi, \tau) \qquad as \quad \epsilon \to 0. \tag{7.9}
$$

We need an auxiliary model of a game with a penalty β, $\beta > 0$. To each pair of controls $\hat{y}(t) = (y(t), b(t))$, $\hat{z}(t)$ as in Section 1 we correspond a payoff

$$P_\beta(\hat{y}, \hat{z}) = \int_\tau^{T_0} \left\{ h(x, t, y, z) + \frac{1}{\beta} b(t)\, g_1(x, t) \right\} \cdot \left\{ \exp \left\{ \int_\tau^t \left[k(x, s, y, z) - \frac{b(s)}{\beta} \right] ds \right\} \right\} dt + g_1(x(T_0), T_0) \exp \left\{ \int_\tau^{T_0} \left[k(x, t, y, z) - \frac{b(t)}{\beta} \right] dt \right\}. \tag{7.10}$$

Consider the differential game of fixed duration associated with (1.1), (1.2), (7.10), Denote its upper δ-value by $V_\beta{}^\delta$. Thus,

$$V_\beta{}^\delta = \sup_{\hat{\Gamma}^\delta} \inf_{\hat{\Delta}_\delta} P_\beta[\hat{\Delta}_\delta, \hat{\Gamma}^\delta] = \inf_{\hat{\Delta}_\delta} \sup_{\hat{\Gamma}^\delta} P_\beta[\hat{\Delta}_\delta, \hat{\Gamma}^\delta]. \tag{7.11}$$

Notice that here $\hat{\Delta}_\delta(\hat{y})$ is a control function, $z(t)$ (not $\hat{z}(t) = (z(t), c(t))$).

THEOREM 7.2. *Under the conditions of Theorem* 7.1,

$$| V_\beta{}^\delta - V^\delta | \leqslant C\beta \tag{7.12}$$

where C is a positive constant independent of δ, β.

Proof. For simplicity we take $\tau = 0$. We also write $g_1(t) = g_1(x(t), t)$. Denote by $P(\hat{y}, \hat{z})$ the payoff given by (1.4). We first prove the inequality

$$V^\delta \geqslant V_\beta{}^\delta - C\beta. \tag{7.13}$$

For any $\alpha > 0$ there is a lower δ-strategy $\hat{\Delta}_\delta{}^\alpha$ such that

$$V^\delta \geqslant \sup_{\hat{\Gamma}^\delta} P[\hat{\Delta}_\delta{}^\alpha, \hat{\Gamma}^\delta] - \alpha.$$

Therefore,

$$V^\delta \geqslant P(\hat{y}, \hat{\Delta}_\delta{}^\alpha(\hat{y})) - \alpha \qquad \text{for any control } \hat{y}. \tag{7.14}$$

We modify $\hat{\Delta}_\delta{}^\alpha$ into $\tilde{\Delta}_\delta{}^\alpha$ as follows.

Denote by S the stopping times corresponding to $\hat{y}$. Then, $\tilde{\Delta}_\delta{}^\alpha(\hat{y})$ coincides with $\hat{\Delta}_\delta{}^\alpha(\hat{y})$ if $t < S + \delta$ and is a constant function $\bar{z}$ if $t > S + \delta$.

$$V^\delta \geqslant P(\hat{y}, \tilde{\Delta}_\delta{}^\alpha(\hat{y})) - \alpha \qquad \text{for any } \hat{y}(\cdot). \tag{7.15}$$

Let us introduce the control $\hat{y}_1(\cdot)$ which is a modification of $\hat{y}(\cdot)$ obtained by taking $b_{1l} = 1$ if $l \geqslant j = 1 + S/\delta$; thus $\hat{y}_1(\cdot) = (y_1(\cdot), b_1(\cdot))$, where $y_1(\cdot) = y(\cdot)$ and $b_1(\cdot)$ has the components

$$b_{1l} = 0 \qquad \text{if} \quad l < j,$$

$$b_{1l} = 1 \qquad \text{if} \quad l \geqslant j.$$

Notice that $\hat{y}(\cdot)$ and $\hat{y}_1(\cdot)$ have the same stopping time S.

Set

$$\hat{\Delta}_\delta^\alpha(\hat{y}_1) = \hat{\Delta}_\delta^\alpha(\hat{y}).$$

Clearly

$$P(\hat{y}, \hat{\Delta}_\delta^\alpha(\hat{y})) = P(\hat{y}_1, \hat{\Delta}_\delta^\alpha(\hat{y}_1)). \tag{7.16}$$

Consider now

$$\begin{aligned} P_\beta(\hat{y}_1, \hat{\Delta}_\delta^\alpha(\hat{y}_1)) &= \int_0^S h \left[\exp \int_0^t kds\right] dt + \int_S^{T_0} \left(h + \frac{g_1}{\beta}\right) \left[\exp \int_0^t kds\right] \left[\exp\left(-\frac{t-S}{\beta}\right)\right] dt \\ &\quad + g_1(T_0) \left[\exp \int_0^{T_0} kds\right] \left[\exp\left(-\frac{T_0 - S}{\beta}\right)\right] \end{aligned}$$

and

$$P(\hat{y}_1, \hat{\Delta}_\delta^\alpha(\hat{y}_1)) = \int_0^S h \left[\exp \int_0^t kds\right] dt + g_1(S) \left[\exp \int_0^S kdt\right].$$

Clearly

$$\begin{aligned} P_\beta(\hat{y}_1, \hat{\Delta}_\delta^\alpha(\hat{y}_1)) - P(\hat{y}_1, \hat{\Delta}_\delta^\alpha(\hat{y}_1)) &= \int_S^{T_0} h \left[\exp \int_0^t kds\right] \exp\left(-\frac{t-S}{\beta}\right) dt \\ &\quad + \int_S^{T_0} \frac{g_1}{\beta} \left[\exp \int_0^t kds\right] \left[\exp\left(-\frac{t-S}{\beta}\right)\right] dt \\ &\quad + g_1(T_0) \left[\exp \int_0^{T_0} kds\right] \exp\left(-\frac{T_0 - S}{\beta}\right) - g_1(S) \exp \int_0^S k\, dt, \end{aligned} \tag{7.17}$$

and, by integration by parts,

$$= \int_S^{T_0} (h + g_1' + kg_1) \left[\exp \int_0^t kds\right] \left[\exp\left(-\frac{t-S}{\beta}\right)\right] dt.$$

Since

$$| h + g_1' + kg_1 | \left[\exp \int_0^t kds\right] \leqslant C, \tag{7.18}$$

C depending on initial conditions but not on the controls $y(\cdot)$, $z(\cdot)$), and since

$$\int_S^{T_0} \exp\left(-\frac{t-S}{\beta}\right) dt = \beta\left[1 - \exp\left(-\frac{T_0 - S}{\beta}\right)\right],$$

we get

$$P(\hat{y}_1, \hat{\Delta}_\delta^\alpha(\hat{y}_1)) \geqslant P_\beta(\hat{y}_1, \hat{\Delta}_\delta^\alpha(\hat{y}_1)) - C\beta.$$

From (7.17) and (7.18) it then follows that

$$V^\delta \geqslant P_\beta(\hat{y}_1, \hat{\Delta}_\delta^\alpha(\hat{y}_1)) - C\beta - \alpha. \tag{7.19}$$

Now

$$\begin{aligned} P_\beta(\hat{y}, \hat{\Delta}_\delta^\alpha(\hat{y})) &= \int_0^S h\left[\exp\int_0^t kds\right] dt \\ &\quad + \int_S^{T_0} \left(h + \frac{g_1 b}{\beta}\right)\left[\exp\int_0^t kds\right]\left[\exp\left(-\int_S^t \frac{bds}{\beta}\right)\right] dt \\ &\quad + g_1(T_0)\left[\exp\int_0^{T_0} kds\right]\left[\exp - \int_S^{T_0}\frac{bds}{\beta}\right)\right] \\ &= \int_0^S h\left[\exp\int_0^t kds\right] dt + g_1(S)\left[\exp\int_0^S kds\right] \\ &\quad + \int_S^{T_0} (h + g_1' + kg_1)\left[\exp\int_0^t kds\right]\left[\exp\left(-\int_S^t \frac{bds}{\beta}\right)\right] dt. \end{aligned} \tag{7.20}$$

Similarly

$$\begin{aligned} &P_\beta(\hat{y}_1, \hat{\Delta}_\delta^\alpha(\hat{y}_1) \\ &\quad = \int_0^S h\left[\exp\int_0^t kds\right] dt + g_1(S)\left[\exp\int_0^S kds\right] \\ &\qquad + \int_0^{T_0} (h + g_1' + kg_1)\left[\exp\int_0^t kds\right]\left[\exp\left(-\frac{t-S}{\beta}\right)\right] dt, \end{aligned} \tag{7.21}$$

From the assumptions (7.8), we get

$$P_\beta(\hat{y}_1, \hat{\Delta}_\delta^\alpha(\hat{y}_1)) \geqslant P_\beta(\hat{y}, \hat{\Delta}_\delta^\alpha(\hat{y}))$$

and thus from (7.19) it follows

$$V^\delta \geqslant P_\beta(\hat{y}, \hat{\Delta}_\delta^\alpha(\hat{y})) - C\beta - \alpha. \tag{7.22}$$

This implies (7.13).

Let us now prove that

$$V^\delta \leqslant V_\beta{}^\delta + C\beta. \tag{7.23}$$

For any $\alpha > 0$, there exists $\hat{\Gamma}_\alpha{}^\delta$ such that

$$V^\delta \leqslant P(\hat{z}, \hat{\Gamma}_\alpha{}^\delta(\hat{z})) + \alpha, \qquad \text{for every } \hat{z}. \tag{7.24}$$

We shall modify $\hat{\Gamma}_\alpha{}^\delta$ into $\hat{\hat{\Gamma}}_\alpha{}^\delta$ as follows: denoting by S_α the stopping time for $\hat{\Gamma}_\alpha{}^\delta(\hat{z})$,

if $S_\alpha = (j-1)\,\delta$, then we take $b_l = 1$, $l \geqslant j+1$, without changing the $y(\cdot)$ (notice that $b_j = 1$);

We have

$$P(z, \hat{\Gamma}_\alpha{}^\delta(z)) = P(z, \hat{\hat{\Gamma}}_\alpha{}^\delta(z)), \qquad \text{for every } z,$$

hence from (7.24)

$$V^\delta \leqslant P(z, \hat{\hat{\Gamma}}_\alpha{}^\delta(z)) + \alpha \qquad \text{for every } z. \tag{7.25}$$

Proceeding analogously to the derivation of (7.19), we find that

$$V^\delta \leqslant P_\beta(z, \hat{\hat{\Gamma}}_\alpha{}^\delta(z)) + C\beta + \alpha,$$

and (7.23) follows.

Let us now consider $u_{\epsilon\beta}$, the approximation of the solution of u_ϵ of (7.7) obtained by penalizing the constraint, i.e., $u_{\epsilon\beta}$ is the unique solution of

$$\begin{aligned}
&u_{\epsilon\beta} \in L^p(0, T_0\,;\, W^{2,p,\mu}) \cap L^2(0, T_0\,;\, W^{2,2,\mu}) \cap L^\infty(0, T_0\,;\, W^{1,2,\mu}),\\
&\partial u_{\epsilon\beta}/\partial t \in L^p(0, T_0\,;\, W^{0,p,\mu}) \cap L^2(0, T_0\,;\, W^{0,2,\mu}),\\
&(\partial u_{\epsilon\beta}/\partial t) + (\epsilon^2/2)\,\Delta u_{\epsilon\beta} + H^+(x, t, u_{\epsilon\beta}, \partial u_{\epsilon\beta}/\partial x) + 1/\beta\,(u_{\epsilon\beta} - g_1)^- = 0,\\
&u_{\epsilon\beta}(T_0) = g_1(T_0).
\end{aligned} \tag{7.26}$$

Notice that $u_{\epsilon\beta}$, $\partial u_{\epsilon\beta}/\partial x$ are bounded functions in compact subsets (cf. Remark 1 at the end of Section 4.) From the $(1+\delta)$-estimate and from the Schauder type interior estimates for parabolic equations (see Friedman [9]) it then follows that, for any $R > 0$, $\delta > 0$,

$$\begin{aligned}
&| Z(x, t) - Z(\bar{x}, \bar{t})| \leqslant C(| t - \bar{t} |^{\alpha/2} + | x - \bar{x} |^{\alpha/2})\\
&\quad \text{for } Z = u_{\epsilon\beta},\ \partial u_{\epsilon\beta}/\partial x,\ \partial u_{\epsilon\beta}/\partial t,\ \partial^2 u_{\epsilon\beta}/\partial x^2, \text{ provided } | x | < R,\ | \bar{x} | < R,\\
&\qquad 0 \leqslant t, \bar{t} \leqslant T_0 - \delta,
\end{aligned} \tag{7.27}$$

where $C = C(\epsilon, \beta)$, $0 < \alpha < 1$, $\alpha = \alpha(\epsilon, \beta)$.

Now from Elliott–Kalton [4], Friedman [13], [16], $u_{\epsilon\beta}$ can be considered as the value of a certain stochastic differential game without stopping times (not in the sense of Section 5). In that context, it is important to introduce the sequence $V^{\delta}_{\epsilon\beta}$ which plays for the stochastic differential game the same role as $V_{\beta}{}^{\delta}$ does for the deterministic game with payoff (7.10). Since it would be too lengthy to describe completely the stochastic differential game for which $u_{\epsilon\beta}$ is the value, we shall only present those results which we will use later

Let

$$t_j = \delta j.$$

For a starting point (ξ, t_j) one defines two sets:

$\mathscr{D}_j$ which is a set of stochastic strategies for the minimizer, and

$\mathscr{H}_j$ which is a set of stochastic controls for the maximizer.

Any element D of $\mathscr{D}_j$ together with a control $\hat{y}$ in $\mathscr{H}_j$ determine a control z for the minimizer. Now, $\hat{y}$ and z are actually a pair (y, b) and z as we know it in the deterministic case, but depending on a stochastic parameter. The first result we will use is that

$$V^{\delta}_{\epsilon\beta}(t_j, \xi) = \inf_{D\in\mathscr{D}_j} \sup_{\hat{y}\in\mathscr{H}_j} P^{\epsilon}_{\beta j}(\xi; \hat{y}, z) \tag{7.28}$$

where z in (7.28) is determined by D and $\hat{y}$, and

$$\begin{aligned} P^{\epsilon}_{\beta j}(\xi; \hat{y}, z) = E \int_{t_j}^{T_0} \Big\{ &\Big[h(x_{\epsilon}(t; \omega), t, y(t; \omega), z(t; \omega)) \\ &+ \frac{1}{\beta} b(t; \omega)\, g_1\Big] \Big[\exp \int_{t_j}^{t} \Big(k - \frac{b}{\beta}\Big)\, ds\Big]\Big\}\, dt \\ &+ Eg(x_{\epsilon}(T_0; \omega), T_0) \\ &\times \Big[\exp \int_{t_j}^{T_0} \Big[k(t, x_{\epsilon}(t; \omega), y(t; \omega), z(t; \omega) - \frac{b}{\beta}\Big]\, ds\Big] \end{aligned} \tag{7.29}$$

where $x_{\epsilon}(t; \omega)$ is a stochastic process defined by y, z (independent of b) and Friedman's stochastic model [16].

Furthermore

$$\begin{aligned} V^{\delta}_{\epsilon\beta}(\xi, t_j) = \inf_{z_j} \sup_{y_j; b_j} E \Big\{ &V^{\delta}_{\epsilon\beta}(x_{\epsilon}(t_{j+1}), t_{j+1}), \Big[\exp \int_{t_j}^{t_{j+1}} \Big(k - \frac{b_j}{\beta}\Big)\, ds\Big] \\ &+ \int_{t_j}^{t_{j+1}} \Big[h + \frac{1}{\beta} b_j g_1\Big] \Big[\exp \int_{t_j}^{t} \Big(k - \frac{b}{\beta}\Big)\, ds\Big]\, dt \Big\} \end{aligned} \tag{7.30}$$

where z_j and y_j are any measurable mappings from (t_j, t_{j+1}) into Z and Y respectively and b_j are identically equal to 0 or 1.

The second important feature is that $V_\beta^\delta(t_j, \xi) = V_{0\beta}^\delta(t_j, \xi)$. This is just a consequence of the fact that the relations (7.30) make sense for $\epsilon = 0$ and then coincide with the recurrence relationships satisfied by $V_\beta^\delta(\xi, t_j)$.

We have

LEMMA 7.1.

$$| V_{\epsilon\beta}^\delta(\xi, t_j) - V_\beta^\delta(\xi, t_j)| \leqslant C\epsilon \tag{7.31}$$

where C does not depend on j, ξ, δ and β.

Proof. Since from (7.28) and $V_\beta^\delta = V_{0\beta}^\delta$ we have

$$V_\beta^\delta(\xi, t_j) = \inf_{D\in\mathscr{D}_j} \sup_{\hat{y}\in\mathscr{H}_j} P_{\beta j}^0(\xi; \hat{y}, z),$$

it is enough to compare $P_{\beta j}^\epsilon$ and $P_{\beta j}^0$ for fixed $\hat{y}$, z, provided we get an estimate uniform in $\hat{y}$, z. But

$$\begin{aligned}
&P_{\beta j}^\epsilon(\xi; \hat{y}, z) - P_{\beta j}^0(\xi; \hat{y}, z)\\
&= \int_{t_j}^{T_0} \Big\{ h(x_\epsilon, t, y, z) \left[\exp \int_{t_j}^t \left(k_\epsilon(s) - \frac{b}{\beta}\right) ds\right]\\
&\quad - h(x, t, y, z) \left[\exp \int_{t_j}^t \left(k(s) - \frac{b}{\beta}\right) ds\right]\Big\}\, dt\\
&\quad + \int_{t_j}^{T_0} \frac{bg_1(x_\epsilon(t), t)}{\beta} \left[\exp \int_{t_j}^t \left(k_\epsilon(s) - \frac{b}{\beta}\right) ds\right]\\
&\quad - \frac{bg_1(x(t), t)}{\beta} \left[\exp \int_{t_j}^t \left(k(s) - \frac{b}{\beta}\right) ds\right]\Big\}\, dt\\
&\quad + g_1(x_\epsilon(T_0), T_0) \left[\exp \int_{t_j}^{T_0} \left(k_\epsilon(s) - \frac{b}{\beta}\right) ds\right]\\
&\quad - g_1(x(T_0), T_0) \left[\exp \int_{t_j}^{T_0} \left(k(s) - \frac{b}{\beta}\right) ds\right],
\end{aligned} \tag{7.32}$$

where we have written $k_\epsilon(s)$ for $k(x_\epsilon(s), s, y(s), z(s))$.

Since

$$\Big| h(x_\epsilon, t, y, z) \Big[\exp \int_{t_j}^t k_\epsilon(s)\, ds\Big]$$

$$- h(x, t, y, z) \Big[\exp \int_{t_j}^t k_\epsilon(s)\, ds\Big] \exp \Big[- \int_{t_j}^t \frac{b}{\beta}\Big] \Big|$$

$$\leqslant \Big| h(x_\epsilon, t, y, z) \exp \Big[\int_{t_j}^t k_\epsilon(s)\, ds\Big] - h(x, t, y, z) \exp \Big[\int_{t_j}^t k_\epsilon(s)\, ds\Big] \Big|,$$

we get for the first integral on the right-hand side of (7.32) an estimate independent of β, which can be treated by the same methods as in Friedman [16]. The same is true for the last term

$$g_1(x_\epsilon(T_0), T_0) \Big[\exp \Big[\int_{t_j}^{T_0} \Big(k_\epsilon(s) - \frac{b}{\beta}\Big)\, ds\Big]\Big]$$

$$- g_1(x(T_0), T_0) \Big[\exp \Big[\int_{t_j}^{T_0} \Big(k(s) - \frac{b}{\beta}\Big)\, ds\Big]\Big].$$

Next we have

$$E \int_{t_j}^{T_0} \Big| g_1(x_\epsilon(t), t) \Big[\exp \int_{t_j}^t k_\epsilon(s)\, ds\Big] - g_1(x(t), t) \Big[\exp \int_{t_j}^t k(s)\, ds\Big] \Big|$$

$$\cdot \frac{b}{\beta} \exp \Big(- \int_{t_j}^t \frac{b}{\beta}\, ds\Big)\, dt \leqslant C\epsilon \int_{t_j}^{T_0} \frac{b}{\beta} \Big[\exp \Big(- \int_{t_j}^t \frac{b}{\beta}\, ds\Big)\Big]\, dt, \tag{7.33}$$

which comes from

$$E \mid x_\epsilon(t) - x(t)|^2 \leqslant C\epsilon^2$$

(cf. [16]), and the Lipschitz properties of g_1 and k. Since

$$\int_{t_j}^{T_0} \frac{b}{\beta} \Big[\exp \Big(- \int_{t_j}^t \frac{b}{\beta}\, ds\Big)\Big]\, dt \leqslant 1,$$

we can estimate the right-hand side of (7.33) by $C\epsilon$.

This completes the proof of the Lemma.

We shall now prove Theorem 7.1. For simplicity we shall establish (7.9) for $\tau = 0$.

From Theorem 7.2, we know that

$$| V^\delta - V_\beta^\delta | \leqslant C\beta. \tag{7.34}$$

From the general results of differential games without stopping times (see [4], [13] and [16]), we have

$$| V^{\delta}_{\epsilon\beta} - u_{\epsilon\beta} | \leqslant M_1(\beta, \epsilon) 0_{\beta\epsilon}(\delta) \tag{7.35}$$

where $0_{\beta\epsilon}(\delta) \to 0$ when $\delta = (T_0/2^n) \to 0$ (for fixed β, ϵ).

From the proof of Theorem 4.2 and the fact that $p > m$ we know that

$$u_{\epsilon\beta} \to u_\epsilon \quad \text{for all} \quad (x, t), \quad \text{as} \quad \beta \to 0.$$

Therefore,

$$| u_\epsilon - u_{\epsilon\beta} | \leqslant \tilde{0}(\epsilon, \beta) \tag{7.36}$$

where

$$\tilde{0}(\epsilon, \beta) \to 0 \quad \text{when} \quad \beta \to 0, \quad \text{for fixed } \epsilon.$$

From (7.34), (7.35), (7.36) and (7.31) it follows that

$$| V^{\delta} - u_\epsilon | \leqslant C\beta + C\epsilon + M_1(\beta, \epsilon) 0_{\beta\epsilon}(\delta) + \tilde{0}(\epsilon, \beta), \tag{7.37}$$

By letting $\delta \to 0$, we get

$$| V^{+} - u_\epsilon | \leqslant C\beta + C\epsilon + \tilde{0}(\epsilon, \beta). \tag{7.38}$$

Therefore, by letting $\beta \to 0$, we get

$$| V^{+} - u_\epsilon | \leqslant C\epsilon$$

which ends the proof of Theorem 7.1.

In Section 1 we have defined V^+ as $\lim V^\delta$ where $\delta = (T - \tau)/2^n$. The definition of V^δ makes sense for any $\delta = (T - \tau)/N$, where N is a positive integer. The proof of Theorem 7.1 shows that

$$\limsup V^\delta \leqslant u_\epsilon + C\epsilon,$$

$$\liminf V_\delta \geqslant u_\epsilon - C\epsilon$$

where $\delta = (T - \tau)/N$, $N = 1, 2, \ldots$. Hence:

COROLLARY 7.3. *Under the assumptions of Theorem* 7.1, $\lim V^\delta$ *exists as* $\delta = (T - \tau)/N$, $N = 1, 2, \ldots, N \to \infty$.

Remark. Given a general function F as in Section 3, consider the nonlinear parabolic variational inequality as in Theorem 4.2, with

$L = \epsilon^2/2\Delta$, $g_2 \equiv \infty$. Denote the solution by W_ϵ. Suppose that F satisfies also (3.12) and the conditions:

$$|F_x| + |F_t| \leqslant C(1 + |p|).$$

Then f, k, h (as defined in Section 3) satisfy the conditions in (7.2). Suppose also that g_1 is a function satisfying the conditions in (7.2), (7.6), (7.8). Then Theorem 7.1 can be applied. It follows that

$$\lim_{\epsilon \to 0} W^\epsilon(x, t) \quad \text{exists.}$$

Further, we can characterize the limit V^+ as the upper value of a certain game with dynamics, payoff and control sets which depend on F. Finally, V^+ satisfies the limiting nonlinear variational inequality.

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